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Abstract. We are interested in proving that if we use CS iterations of forcing notions not adding reals that satisfies additional conditions then the limit forcing does not add reals. As a result we prove that we can amalgamate two earlier methods and prove the consistency with ZFC + G.C.H. of two statements gotten separately earlier: SH and non-club guessing. We also prove the consistency of further cases of "strong failure of club guessing".

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Anotated Content

§0 Introduction

[We give lengthy explanations of the problems and proofs for No-New-Reals iterations (or proper forcing, CS iterations).]

§1 Preservation of not adding reals

[We present sufficient conditions for CS iteration of proper forcing not to add reals. For this we define "reasonable parameters $\mathfrak{p}$" and we have two main demands. One (clause (c) of Definition 1.8) is a weakening of "α-proper for every $\alpha < \omega_1$". This time it has the form (on Q_i), $\mathfrak{p}$-proper which informally says that: if $\mathfrak{p} \in N, Y \subseteq \{M \in N : M$ appropriate$\}$ is α-large then for some (N, Q_i)-generic condition $q \geq p, q$ forces that $\{M \in Y : M[\underset{\sim}{G}_Q] \cap V = M\}$ is α-large (the meaning of $\alpha-$ large depends on $\mathfrak{p}$), hence without loss of generality, $\mathfrak{p}$ has length ω_1. The other main demand (clause (d) of Definition 1.8) is a "weak diamond preventive".
We then show that α-properness for $\alpha < \omega_1$ is sufficient for the first main demand (in 1.16(3)). The demand on the games for $\mathfrak{p}$ helps to prove the preservation of $\mathfrak{p}$-properness.]

§2 Delayed properness

[The preservation theorem in the first section does not, for standard $\mathfrak{p}$, cover shooting a club $C \subseteq \omega_1$ running away for $C_\delta \subseteq \delta = \sup(C_\delta), C_\delta$ small (see §3). For this we will use $(\mathfrak{p}, \alpha, \beta)$-proper for enough pairs $\alpha \leq \beta < \ell g(\mathfrak{p})$ (so starting from β-large we get α-large; for many α we can choose $\beta = \alpha$ but during the inductive proof we pass through cases of $\alpha < \beta$).
Here we introduce various definitions and basic facts needed. We discuss axioms, version of the properties preserved by CS iterations and strengthening of the iteration Lemmas of §1.]

§3 Example: shooting a thin club

[We present the natural forcing showing $\kappa = 2$ is interesting (not only $\kappa = \aleph_0$) (from [Sh:b, Ch.VIII,§4]).
We show that the natural forcing (see above) for running away from $C_\delta \subseteq \delta$, of small order type (see [Sh:f, Ch.XVIII,§2]) falls under our framework for delayed properness. We give examples: running away from $\langle C_{\delta,0}, C_{\delta,1} : \delta < \omega_1 \text{ limit}\rangle, C_{\delta,0}, C_{\delta,1}$ are disjoint closed subsets of δ with no restrictions on their order type so we ask for $C, C \cap C_{\delta,0}$ or $C \cap C_{\delta,1}$ is bounded in δ and more.]

§4 Second preservation of not adding reals

[We give a sufficient condition for the limit not to add reals. We here are weakening the demand "$\mathfrak{p}$-proper", using $(\mathfrak{p}, \alpha, f(\alpha))$-proper instead $(\mathfrak{p}, \alpha, \alpha)$-proper, what we called delayed properness. The price is that here $\mathfrak{p}$ has length of large cofinality, so essentially we catch our tails on a club of it. Also the Lemma here covers the examples.]

§5 Problematic Forcing

[We discuss further generalizations.]

§0 Introduction

We try to explain our problems and results. If the explanations look opaque, try to return to them after reading at least part of the proof. Sections §0, §1 are based on lectures in the logic seminar in the Hebrew University, Spring 1997, whose participants I thank. On the history see in [Sh:f, V,§7,VIII,§4,XVIII,§1,§2], [Sh 666, §3].

0.1 Definition. 1) Let K_0 be the family of CS iterations, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$, we denote $P_\alpha = \mathrm{Lim}(\bar{Q})$.
2) We say $\bar{Q}$ is proper if each Q_i is (which means: every $f \in ({}^\omega\omega)^{V^{P_{i+1}}} = ({}^\omega\omega)^{(V^{P_i})^{Q_i}}$ is bounded by some $g \in ({}^\omega\omega)^{V^{P_i}}$, hence P_j/P_i is proper for $i < j \le \alpha$ (see [Sh:b] or [Sh:f]).
3) We say $\bar{Q}$ is ${}^\omega\omega$-bounding if each Q_i is[1], hence P_j/P_i is ${}^\omega\omega$-bounding for $i < j \le \alpha$.
4) We say $\bar{Q}$ is NNR if $i < \alpha \Rightarrow P_{i+1}$ adds no reals.
[Equivalently: $i < \alpha \Rightarrow Q_i$ adds no reals and $\beta < \alpha \Rightarrow P_\beta$ adds not reals.]

Discussion: It would be nice if also this (NNR) would be preserved in limit. But this is wrong for two known reasons, obstacles, explained below:

$\bigotimes_1$ weak diamond
$\bigotimes_2$ existence of clubs.

Our aim here is to weaken the medicine one uses against $\bigotimes_2$ for CS iterations of proper forcing notions. Let us explain the "obstacles".

Concerning the weak diamond:

Let $\bar{\eta} = \langle \eta_\delta : \delta < \omega_1, \delta \text{ limit} \rangle, \eta_\delta = \langle \eta_\delta(n) : n < \omega \rangle$ where η_δ is an ω-sequence of ordinals (strictly) increasing with limit δ. Let D be a non-principal ultrafilter on ω.
For $f \in {}^{\omega_1}2, \delta < \omega_1$ limit, let $\mathrm{Av}_D(f, \eta_\delta) = \ell$ iff $\{n : f(\eta_\delta(n)) = \ell\} \in D$.

0.2 Question[CH]: Given $\bar{e} = \langle e_\delta : \delta < \omega_1 \text{ limit} \rangle, e_\delta \in \{0,1\}$ is there $f \in {}^{\omega_1}2$ such that for a club of $\delta < \omega_1$ we have $e_\delta = \mathrm{Av}_D(f, \eta_D)$?

Naturally, trying to prove consistency we should use a CS iteration $\bar{Q}$, for simplicity we assume

[1]where Q is ${}^\omega\omega$ bounded (in the universe V) if every $f \in ({}^\omega)^{V^Q}$ is bounded by some $g \in ({}^\omega\omega)^V$

$V \models GCH, \bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \omega_2 \rangle$

$$Q_i = Q_{\bar{e}} \text{ (i.e. } \underset{\sim}{Q}_i = Q_{\underset{\sim}{\bar{e}}}) \text{ where}$$

$$Q_{\bar{e}} = \Big\{ f : \text{for some } \zeta < \omega_1, f \in {}^{\zeta}2 \text{ and for every limit } \delta \le \zeta \text{ we have } \mathrm{Av}_D(f, \eta_\delta) = e_\delta \Big\}.$$

This is a very nice forcing notion - it is proper (even $< \omega_1$-proper, see below) and NNR and for every $\alpha < \omega_1, \mathscr{I}_\alpha = \{f \in Q_{\bar{e}} : \alpha \subseteq \mathrm{Dom}(f)\}$ is a dense open subset of it.
But the weak diamond ([DvSh 65] or see [Sh:b, Ch.XII,§1] = [Sh:f, AP,§1]) tells us for this case that the answer is <u>no</u>, that is:
$\exists \bar{e} \forall f \in {}^{\omega_1}2 \exists^{\text{stat}} \delta(e_\delta \ne \mathrm{Av}_D(f, \eta_\delta))$.

In fact this holds for any function $\mathrm{Av}' : \bigcup\limits_{\delta < \omega_1} {}^{\delta}2 \to \{0,1\}$.

<u>Why is $Q_{\bar{e}}$ NNR</u>?

Let $\langle N_i : i \le \omega^2 \rangle$ be increasing continuously with $i, Q_{\bar{e}} \in N_0$ where $N_i \prec (\mathscr{H}(\chi), \in)$ is countable, $\bar{N} \restriction (i+1) \in N_{i+1}$; let $\delta_i = N_i \cap \omega_1$. So $\langle \delta_i : i \le \omega^2 \rangle$ is strictly increasing continuously. But $\{\eta_{\delta_{\omega^2}}(n) : n < \omega\} \subset \delta_{\omega^2}$ has order type ω, so $W = \{i < \omega^2 : \exists n(\delta_i \le \eta_{\delta_{\omega^2}}(n) < \delta_{i+1})\}$ has order type ω.

So we can find $\ell_n < \omega$ such that $\bigwedge\limits_{n<\omega} \bigwedge\limits_{m<\omega} \omega \times n + \ell_n + m \notin W$. Let $p \in Q_{\bar{e}} \cap N_0$, let $\underset{\sim}{\tau} : \omega \to \mathrm{Ord}, \underset{\sim}{\tau} \in N_0$. We choose by induction on $n, p_n \in Q_{\bar{e}}$ such that $p \le p_n, p_{n-1} \le p_n, p_n \in N_{\omega n + \ell_n + 1}, p_n$ force some value to $\underset{\sim}{\tau}(n)$ and on $[\delta_{\omega m + \ell_m}, \delta_{\omega(m+1)+\ell_{m+1}}) \cap \mathrm{Rang}(\eta_\delta) \setminus \mathrm{Dom}(p)$ which agrees with $e_{\delta_{\omega^2}}$.
So

(∗) the desired demand on $\bar{Q}_i$ (guaranteeing P_α in NNR) should exclude the $Q_{\bar{e}}$'s.

0.3 Definition. Let K_1 be the class of proper ${}^{\omega}\omega$-bounding iteration $\bar{Q} \in K_0$, let K_2 be the class of NNR iterations $\bar{Q} \in K_1$, and let K_3 be the class of $\bar{Q} \in K_2$ such that

(∗) if χ is large enough (so $\bar{a} \in \mathscr{H}(\chi)$),
$N \prec (\mathscr{H}(\chi), \in)$ countable,

$\bar{Q} \in N$,
$i \in \ell g(\bar{Q}) \cap N$,
$p \in P_{i+1} \cap N$,
$q_0, q_1 \in P_i$ are (N, P_i)-generic
(i.e. $q_\ell \Vdash N[\underset{\sim}{G}_{P_i}] \cap V = N$) and moreover
$q_\ell \Vdash$ "$\underset{\sim}{G}_{P_i} \cap N = G^*$" and
$p \restriction i \le q_\ell$
<u>then</u> we can find q'_0, q'_1, G^{**} such that $p \restriction (i+1), q_\ell \le q'_\ell \in P_{i+1}$,
$q'_\ell \Vdash$ "$\underset{\sim}{G}_P \cap N = G^{**}$",
q'_ℓ is (N, P_{i+1})-generic, so $G^{**} \subseteq P_{i+1} \cap N$ is generic over N.

<u>This tries to say</u>:

We know $\underset{\sim}{G}_{P_i} \cap N$ (as being G^*) and we are looking at $N[G^*]$ (formally, only its isomorphism type). So we know $\underset{\sim}{Q}^N_i[G^*_i]$.

We would like to find $G' \subseteq \underset{\sim}{Q}^N_i[G^*]$ generic over $N[G^*]$, so that G^*, G' will determine G^{**}. But we need a guarantee that G' will have an upper bound in $\underset{\sim}{Q}_i[G_{P_i}]$. If we know G_{P_i}, fine; but in a sense, we are given 2 candidates (by q_0, q_1 and can increase then to $q'_0 \restriction i, q'_1 \restriction i$) and have to find G' "accepted" by both.

This (weak diamond) obstacle was overcome, with a price, i.e. more than being in K_3, [Sh:f, V,§7] using $\aleph_1$-completeness systems and [Sh:f, XVIII,§4] for 2-completeness systems (phrased in [Sh:f, XVIII,§2] without them).

Unfortunately, this does not suffice: there is an extreme example where e.g. if for q_0, q_1 incompatible in $Q_i, \underset{\sim}{E}_i$ a Q_i-name of a club, for some $\alpha(q_0, q_1)$ we have:

$$q_\ell \le q'_\ell$$

$$q_\ell \Vdash \text{“}\underset{\sim}{E}_i \cap \delta = E^\delta_i \Rightarrow E^{\delta_0}_i \cap E^{\delta_1}_1 \backslash \alpha(q_0, q_1) \text{ is finite”}.$$

This represents the reason, the obstacle which we shall call $\bigotimes_2$, it is overcomed (with a price) either with $(< \omega_1)$-properness or by a kind of "finite powers are proper" (see below).

0.4 Definition. 1) Q is α-proper when:

if $\bar{N} = \langle N_i : i \le \alpha \rangle$ is $\prec$- increasing continuously, $\alpha \in N_0$
$N_i \prec (\mathscr{H}(\chi), \in)$ countable, $\bar{N} \restriction (i+1) \in N_{i+1}$,
$Q \in N_0$ and $p \in Q \cap N_0$
<u>then</u> there is $q, p \le q \in Q_i$ such that q is (N_i, q)-generic for $i \le \alpha$.

2) A forcing notion Q is $(<^+ \omega_1)$-proper if the above holds for any $\alpha < \omega_1$ even omitting "$\alpha \in N_0$". We say Q is $(< \omega_1)$-proper if Q is α-proper for any $\alpha < \omega_1$. So $(< \omega_1)$-proper is an antidote to such problems, i.e. against "reason $\otimes_2$". Okay for specializing a Aronszajn tree and many others, but it seems to me since [Sh 177] too strong: it kills the following:

0.5 Question: Let $\bar{C} = \langle C_\delta : \delta < \omega_1, \delta \text{ limit}\rangle, C_\delta \subseteq \delta = \sup(C_\delta)$, $\text{otp}(C_\delta) = \omega$ or at least $< \delta$, is there a club E of ω_1 such that $\delta < \omega_1 \Rightarrow \delta > \sup(C_\delta \cap E)$? (i.e. is this consistent with CH).

We consider

$$Q^1_{\bar{C}} = \Big\{ \bar{f} : \text{for some non-limit } \alpha < \omega_1 \text{ we have } f \in {}^\alpha 2, f^{-1}(\{1\}) \text{ closed}$$
$$\text{and } \delta < \alpha \text{ limit } \Rightarrow \sup(f^{-1}(\{1\}) \cap C_\delta) < \delta \Big\}.$$

This is the natural forcing for adding a club such that $\bigwedge_\delta [C_\delta \cap E \text{ bounded in } \delta]$. So E "runs away" from each C_δ. This forcing notion is NOT ω-proper: if $\langle N_i : i \le \omega\rangle$ satisfies $C_{N_\omega \cap \omega_1} = \{N_i \cap \omega_1 : i < \omega\}$, then no $f \in Q$ is (N_i, G)-generic, for infinitely many i's.

A solution ([Sh:f, XVIII,§2]) was to demand "essentially" e.g. $P_i \times P_i$ is proper for $i < \ell g(\bar{Q})$. While this is fine for Q^1_i, this seems to exclude specializing an Aronszajn tree by not adding reals. We will deal with a condition implied by both $(< \omega_1)$-proper and (essentially) "the square of the forcing notion is proper".

Continuing explanation:

So for CS iteration $\bar{Q}$ of proper forcing the "reasons", "dangers" for adding reals may come from:

$\bigotimes_0$ (0-reason, danger) some Q_i adds reals and

$\bigotimes_1$ weak diamond.

Against this, we will assume something like (Definition 0.3): many times in some sense $q_0, q_1 \in P_i$ are (N, P_i)-generic, $p \in \underset{\sim}{Q}_i \cap N, q_\ell \Vdash_{P_i}$ "$\underset{\sim}{G}_{P_i} \cap N = G^*$" and for some $G', q'_0 \ge q_0, q'_1 \ge q_1$ in P_{i+1} we have $G' \subseteq (\underset{\sim}{Q}_i \cap N)[G^*]$ and $q'_\ell \Vdash_{P_i}$ "$\underset{\sim}{G}_{Q_i} \cap N[G^*] = G'$" and $p \in G'$.

It is simpler in the proof to allow $q_\ell (\ell < n)$ for some $n < \omega$; but in addition we have the obstacle:

$\bigotimes_2$ adding almost disjoint clubs ([Sh:f, Ch.XVIII,§1]).

There were two medicines:

(α) α-proper for every $\alpha < \omega_1$
(β) something like $P_i \times P_i$ is proper.

In the proofs we have a situation:

$(*)$ $\bar{Q} \in N_0 \in N$
$N_0 \prec (\mathscr{H}(\chi), \in)$ and $N \prec (\mathscr{H}(\chi), \in)$ are countable
q_ℓ is (N, P_i)-generic and (N_0, P_i)-generic
q_ℓ forces that $\underset{\sim}{G}_{P_i} \cap N = G_\ell$, (for $\ell < 2$)
$G^* = G_1 \cap N_0 = G_2 \cap N_0$
$i, j, p \in N_0[G^*], i \leq j \leq \ell g(\bar{Q})$
$p \in P_j, p \restriction i \in G^*$ (possibly more).

We would like to find $G' \subseteq P_j^N / G^*$ generic over N_0 such that q_0 and q_1 both forces that it has an upper bound in $P_j / \underset{\sim}{G}_{P_i}$. If $j = i + 1$ this means $G' \subseteq \underset{\sim}{Q}_i[G^*]$ is generic over N_0 such that q_0, q_1 both force that G' has an upper bound in $\underset{\sim}{Q}_i[G_{P_i}]$. It is natural to demand $G' \in N$, otherwise the two possible generic extensions (for q_0 and q_1) become not related. For the case $j = i+1$, a "weak diamond medicine" should help us. But we need it for every j, naturally we prove it by induction on j, and the successor case can be reduced to the case $j = i + 1$.

But to continue in a limit we need $G' \in N$ and more: for some intermediate $N_1, N_0 \in N_1 \in N$ also $\bigwedge\limits_\ell [q_\ell \Vdash N_1[\underset{\sim}{G}_{P_i}] \cap V = N_1]$. So the clubs of elementary submodels which q_0, q_1 induce on $\{M \prec N : M \in N\}$ should have non-trivial intersection. This is a major point and it has always appeared in some form. Here the medicine against $\bigotimes_2$ should help, in some way there will be many possible N_1's; but its help has a price: we have to carry it during the induction. On the other hand the models playing the role of N_1 may change, we may "consume it and discard it".

Note that the discussion is on two levels. Necessary limitations of universes with CH on the one hand, <u>and</u> how we try to carry the inductive proof on appropriate iterations on the other hand; the connection though is quite tight.

So we shall try for $j \in \ell g(\bar{Q}) \cap N_0$ to extend the situation with i being replaced by j while G^* is being increased to G^{**}. We shall prove by induction suitable facts,

with G^{**} the object we are really interested in.
We are given $q_1, q_2 \in P_i$ and would like to find suitable $q'_1, q'_2 \in P_j$ such that $q'_\ell \restriction i = q_\ell$ (otherwise in limit why is there an upper bound?)
So the real action occur for j limit, so we choose $\zeta_n \in N \cap [i,j)$ such that $\zeta_0 = i, \zeta_n < \zeta_{n+1}$ (sometimes better to have i and each ζ_n non-limit) and $\bigcup\limits_{n<\omega} \zeta_n = \sup(j \cap N)$.

You can think of:

> in each case of limit j, proving the inductive statement, we choose a "surrogate" for N called N_1, during the induction it serves like N in the limit dealing with $\zeta_0, \zeta_1, \ldots$ using the induction hypothesis on N_1 we get G^{**} which may not be in N_1 but is in N.

So we try to choose by induction on $n, q_{0,n}, q_{1,n}, G^*_n$ such that: $q_{\ell,n} \in P_{\zeta_n}$ is (N, P_{ζ_n})-generic, $q_{\ell,0} = q_\ell, q_{\ell,n+1} \restriction \zeta_n = q_{\ell,n}, G^*_n \in N_1, G^*_n \subseteq P_{\zeta_n} \cap N$ is generic over N and $q_{\ell,n} \Vdash$ "$\underset{\sim}{G}_{P_{\zeta_n}} \cap N = G^*_n$. The construction of the G^*_n should use little information on the actual $q_{\ell,n}$ so that the choices of the G^*_n can be carried say inside N_1 so that $\langle G^*_n : n < \omega \rangle \in N$. In fact several models will play a role like N_1. By the proof of the preservation of ${}^\omega\omega$-bounding we can choose some N_1 and demand "$q_{\ell,n}$ gives to each P_{ζ_n}-name of an ordinal $\underset{\sim}{\tau}_n \in N_1$, only finitely many possibilities".

Now how does $(<\omega_1)$-proper help?
We can assume in the beginning that $\langle N_{1,\gamma} : \gamma \in A \rangle \in N$ is $\prec$-increasing continuously, $N_0 \prec N_{1,\gamma} \prec N, \langle N_{1,\gamma} : \gamma \le \beta \rangle \in N_{\beta+1}$ with $A =: (j+1) \cap N \backslash i$ and assume q_ℓ is $(N_{1,\gamma}, P_i)$-generic for $\gamma \in A$ (similarly for q'_0, q'_1, j in the conclusion) and demand $q_{\ell,n}$ is $(N_{1,\gamma}, P_{\zeta_n})$-generic for $n < \omega$ and $\gamma \in A \backslash \zeta_n$. We are ignoring several points including how the induction change and having $\ell < 2$ rather than $\ell < n(<\omega)$ which complicates life.

How does "$Q \times Q$ is proper" help?
We demand things like "(q_0, q_1) is $(N_1, P_i \times P_i)$-generic" so this gives many common N_1's, but to preserve this we need more complicated situations. Instead of a "tower" of models of countable length, we have a finite tower of models (say of length 5) where on the bottom we are computing $G^{**} \cap P_{\zeta_n}$ and as we go up less and less is demanded.
The medicine in the present work is $\mathfrak{p}$-properness where "Q is $\mathfrak{p}$-proper" say that if Y is a large family of $M \prec N$ and $p \in Q \cap N, Q \in N$ then for some q we have $p \le q$ and q is (N, Q)-generic and $q \Vdash$ "$\{M \in Y : M[\underset{\sim}{G}_Q] \cap V = M\}$ is large". (The idea of the finite tower is retained in the proof). This is quite obvious in hindsight.

Why is it important to be inside N? Otherwise, we could forget about N and we have q_0, q_1. We know they have a common candidate but we need to increase them

to know if and in limit by the regular having bounded in P_i.
"We need a real not a name of a real."

Note that a sufficient condition for $\mathfrak{p}$-properness for Q, if $\mathfrak{p}$ is standard, is homogeneity.

<ins>Notation</ins>: $\mathfrak{p}$ denote a reasonable parameter.

We thank Todd Eisworth for many corrections; he has continued this work.

§1 Preservation of not adding reals

1.1 Definition. We say $\mathfrak{p} = (\bar{\chi}, \bar{R}, \bar{\mathscr{E}}, \bar{D}) = (\bar{\chi}^{\mathfrak{p}}, \bar{R}^{\mathfrak{p}}, \bar{\mathscr{E}}^{\mathfrak{p}}, \bar{D}^{\mathfrak{p}})$ is a reasonable parameter if for some ordinal α^* called $\ell g(\mathfrak{p})$ we have:

(a) $\bar{\chi} = \langle \chi_\alpha : \alpha < \alpha^* \rangle, \chi_\alpha$ a regular cardinal, $\mathscr{H}((\bigcup_{\beta<\alpha} \chi_\beta)^+) \in \mathscr{H}(\chi_\alpha)$

(b) $\bar{R} = \langle R_\alpha : \alpha < \alpha^* \rangle, R_\alpha \in \mathscr{H}(\chi_\alpha)$;
we could have asked "R_α a relation on $\mathscr{H}(\chi_\alpha)$", no real difference for our purpose; in a sense it codes a club of $[\mathscr{H}(\chi_\alpha)]^{\leq \aleph_0}$.

(c) $\bar{\mathscr{E}} = \langle \mathscr{E}_\alpha : \alpha < \alpha^* \rangle$ where $\mathscr{E}_\alpha \subseteq [\mathscr{H}(\chi_\alpha)]^{\leq \aleph_0}$ is stationary

(d) $\bar{D} = \langle D_\alpha : \alpha < \alpha^* \rangle, D_\alpha$ is a function with domain $\mathscr{E}_\alpha, a \in \mathscr{E}_\alpha \Rightarrow D_\alpha(a)$ is a pseudo-filter on a, i.e. $D_\alpha(a)$ is a family of subsets of a closed under supersets, non-empty if $\alpha > 0$
(and let $D^-_\alpha(a) = (D_\alpha(a))^- = \mathscr{P}(a) \backslash D_\alpha(a))$

(e) for $\alpha < \alpha^*$ we let $\mathfrak{p}^{[\alpha]} =: \langle \bar{\chi} \restriction \alpha, \bar{R} \restriction (\alpha+1), \bar{\mathscr{E}} \restriction \alpha, \bar{D} \restriction \alpha \rangle$, so it belongs to $\mathscr{H}(\chi_\alpha)$.
[Why $\bar{R} \restriction (\alpha+1)$? This makes it an easy demand on $\mathscr{E}_\alpha : N \in \mathscr{E}_\alpha \Rightarrow R_\alpha \in N$].

(f) if $a \in \mathscr{E}_\alpha$, <u>then</u> for some countable $N \prec (\mathscr{H}(\chi_\alpha), \in)$ we have:
a is the universe of N, so we may write $D_\alpha(N)$ instead of $D_\alpha(a)$ and $N \in \mathscr{E}_\alpha$ instead of $|N| \in \mathscr{E}_\alpha$

(g) if $\alpha < \alpha^*$ and $N \in \mathscr{E}_\alpha$, then $\mathfrak{p}^{[\alpha]} \in N$

(h) for $N \in \mathscr{E}_\alpha$ and $X \subseteq N$ we have:
$X \in D_\alpha(N)$ iff $(\bigcup_{\beta<\alpha} \mathscr{E}_\beta) \cap X \cap N \in D_\alpha(a)$

(i) if $N \in \mathscr{E}_\alpha, X \in D_\alpha(N), \beta \in \alpha \cap N$ and $y \in N \cap \mathscr{H}(\chi^{\mathfrak{p}}_\beta)$, <u>then</u> for some $M \in \mathscr{E}_\beta \cap X$ we have $X \cap M \in D_\beta(M)$ and $y \in M$.

1.2 Remark. 1) Note that $\langle \mathscr{E}_\alpha : \alpha < \ell g(\mathfrak{p}) \rangle$ are pairwise disjoint by clause (g) (and clause (e)) so $D(N)$ can be well defined as $D_\alpha(N)$ for the unique α such that $N \in \mathscr{E}_\alpha$).
2) Clearly only $D_\alpha(N) \cap \mathscr{P}(\bigcup_{\beta<\alpha} \mathscr{E}_\beta)$ matters.
3) Note that the most natural case is "$D(N)$ is $\{X \subseteq N : X \neq \emptyset \bmod D^{\rm fil}\}$" for some filter $D^{\rm fil}$ on N.
4) Natural cases are: $D_\alpha(a)$ is a filter on a, and $D_\alpha(a) = \{b \subseteq a : a \backslash b \notin D'_\alpha(a)\}$ for a filter $D'_\alpha(a)$ on a; we say D_α is dual to D'_α or $D_\alpha = (D'_\alpha)^+$.

5) We may add in clause (i) of 1.1 that some $X' \in X \cap M$ belongs to $N \cap D_\beta(M)$. No beginning harm but not necessary at present.

1.3 Definition. 1) In 1.1 we say $\bar{D}$ is standard if for every $\alpha < \alpha^*$ and $N \in \mathscr{E}_\alpha$ we have

$$\begin{aligned} D_\alpha(N) = \Big\{ X \subseteq N : & \text{ for every } \gamma \in N \cap \alpha \text{ and} \\ & y \in N \cap \cup\{\mathscr{H}(\chi_\beta) : \beta \in \alpha \cap N\} \text{ for some} \\ & \beta \in N \cap \alpha \backslash \gamma \text{ and} \\ & M \in X \cap \mathscr{E}_\beta \text{ we have } y \in M, \\ & X \cap M \in D_\beta(M) \Big\}. \end{aligned}$$

2) We say $\mathfrak{p}$ is simple if $\alpha \le \beta < \alpha^* \Rightarrow \alpha \le_{\mathfrak{p}} \beta$ where we let $\alpha \le_{\mathfrak{p}} \beta$ be the following partial order on α^* : $\alpha \le_{\mathfrak{p}} \beta$ iff $\alpha \le \beta < \alpha^* = \ell g(\mathfrak{p})$ and we have $N \in \mathscr{E}^\beta \ \& \ \alpha \in N \Rightarrow M =: N \cap \mathscr{H}(\chi^{\mathfrak{p}}_\alpha) \in \mathscr{E}^{\mathfrak{p}}_\alpha$ and $N \in \mathscr{E}^\beta \ \& \ \alpha \in N \ \& \ Y \in D_\beta(N) \Rightarrow Y \cap \bigcup\limits_{\gamma < \alpha} \mathscr{E}^{\mathfrak{p}}_\gamma \in D_\alpha(M)$.
3) We say $\mathfrak{p}$ is standard if ($\mathfrak{p}$ is a reasonable parameter such that) $\bar{D}^{\mathfrak{p}}$ is standard.
4) If $N \prec (\mathscr{H}(\chi), \in)$ and $N \cap \mathscr{H}(\chi^{\mathfrak{p}}_\alpha) \in \mathscr{E}_\alpha$, (hence $\alpha \in N, \mathfrak{p} \restriction \alpha \in N, R^{\mathfrak{p}}_\alpha \in N$), then we let $D_\alpha(N) = D^{\mathfrak{p}}_\alpha(N)$ be $D^{\mathfrak{p}}_\alpha(N \cap \mathscr{H}(\chi^{\mathfrak{p}}_\alpha))$.

1.4 Convention: If $\bar{D} = \bar{D}^{\mathfrak{p}}$ is standard, we may omit it. If $\mathfrak{p}$ clear from the content, we may write $\mathscr{E}_\alpha$ instead $\mathscr{E}^{\mathfrak{p}}_\alpha$.

1.5 Definition. 1) We say that $\mathfrak{p}$ is a winner or a ∂-winner if:

> for every $\alpha < \ell g(\mathfrak{p}), \alpha > 0$ and $N \in \mathscr{E}^{\mathfrak{p}}_\alpha$, in the game $\partial_\alpha(N) = \partial_\alpha(N, \mathfrak{p})$ (defined below)
> the chooser player has a winning strategy where:

2) $\partial_\alpha(N, \mathfrak{p})$ is the following game:

> a play lasts ω moves, in the n-th move
> the challenger choose $X_n \in D_\alpha(N)$ such that $m < n \Rightarrow X_n \subseteq X_m$
> the chooser chooses $M_n \in X_n$ and $Y_n \subseteq M_n \cap X_n, Y_n \in D(M_n) \cap N$
> the challenger chooses $Z_n \subseteq Y_n$ such that $Z_n \in D(M_n)$.

In the end the chooser wins if $\cup\{\{M_n\} \cup Z_n : n < \omega\} \in D_\alpha(N)$.
3) Assume $N \in N' \prec (\mathscr{H}(\chi), \in)$ and $\mathfrak{p} \restriction \alpha \in N'$, and, of course, $N \prec N'$ are countable. The game $\Game'_\alpha(N, N', \mathfrak{p})$ is defined similarly but during the n-th move, we demand that all the chosen objects belong to N' and in the end the chooser also chooses $X'_n \subseteq X_n, X'_n \in D(N) \cap N'$ and the challenger in the next move has to satisfy $X_{n+1} \subseteq X'_n$. Omitting N' we mean: for any such N' the demand holds.
4) $\mathfrak{p}$ is a non-$\Game$-loser if for $\alpha < \ell g(\mathfrak{p}), \alpha > 0, N \in \mathscr{E}_\alpha$ the challenger has no winning strategy in $\Game_\alpha(N, \mathfrak{p})$.
5) "$\Game'_\alpha$-winner" or "non-$\Game'_\alpha$-loser" means we (in part (1) or part (4)) use $\Game'_\alpha(N, \mathfrak{p})$. We say that "the chooser/challenger wins the game $\Game_\alpha(N)$" if he has a winning strategy and so "the chooser/challenger does not win the game $\Game_\alpha(N)$" says the negation. (Similarly for the other games in this paper).
6) Omitting α means for every $\alpha < \ell g(\mathfrak{p})$.

1.6 Observation. 1) If $\mathfrak{p}$ is a reasonable parameter with the standard $\bar{D}$, then $\mathfrak{p}$ is a winner.
2) If $\mathfrak{p}$ is a $\Game_\alpha$-winner then $\mathfrak{p}$ is a $\Game'_\alpha$-winner; if $\mathfrak{p}$ is a $\Game$-winner, then $\mathfrak{p}$ is a $\Game'$-winner; similarly for a non-loser.

Proof. Straight.

1.7 Definition. Assume $\mathfrak{p}$ is a reasonable parameter, $N \in \mathscr{E}^{\mathfrak{p}}_\alpha, y \in N$ and $P \in N$ is a forcing notion. We let $\mathscr{M}_P[\underset{\sim}{G}_P, N, y] =: \{M \in N : P, y \in M$ and $\underset{\sim}{G}_P \cap M$ is a subset of $P \cap M$ generic over $M\}$. If P is clear from the context, we may omit it; note that $\mathscr{M}_P[G, N, y] = \mathscr{M}_P[G \cap N, N, y]$ so we may write $G \cap N$ instead of G. If $y = \emptyset$ we may omit it.

1.8 Definition. We say $\bar{Q}$ is $\mathfrak{p} - NNR^0_{\aleph_0}$ iteration if:

- (a) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < j(*)\rangle$ is a CS iteration of proper forcing notions which belongs to $\mathscr{H}(\chi^{\mathfrak{p}}_0)$, even $\mathscr{P}(P_{j(*)}) \in \mathscr{H}(\chi^{\mathfrak{p}}_0)$
- (b) forcing with $P_{j(*)} = \text{Lim}(\bar{Q})$ does not add reals
- (c) [long properness] if $i \leq j \leq j(*), \alpha < \ell g(\mathfrak{p}), N \in \mathscr{E}_\alpha, \{i, j, \bar{Q}\} \in N$, the condition $q \in P_i$ is (N, P_i)-generic forcing $\underset{\sim}{G}_{P_i} \cap N = G$ and $p \in P_j \cap N, p \restriction i \in G$ and $Y \subseteq \mathscr{M}_{P_i}[G, N, y]$ where $y = \langle \bar{Q}, i, j\rangle$ and $Y \in D_\alpha(N)$ (note that from $\bar{Q}, G$ the ordinal i is reconstructible);
  then there are G'', q' such that:
  - (α) $q' \in P_j, p \leq q'$ and $q \leq q' \restriction i$

(β) q' is (N, P_j)-generic

(γ) q' forces $\underset{\sim}{G}_{P_j} \cap N = G''$

(δ) $Y \cap \mathscr{M}_{P_j}[G'', N, y] \in D_\alpha(N)$

(in §2 this is called $\mathfrak{p}$-proper)

(d) [anti-w.d.] assume $i \le j \le j(*), \alpha < \ell g(\mathfrak{p}), N_0 \in N_1 \in \mathscr{E}_\alpha, N_0 \in \bigcup\limits_{\beta<\alpha} \mathscr{E}_\beta$, $\text{otp}(N_0 \cap [i,j)) < \alpha$ and[2] $n < \omega$, for $\ell < n$ we have $q_\ell \in P_i$ is (N_1, P_i)-generic and q_ℓ forces $\underset{\sim}{G}_{P_i} \cap N_1 = G^\ell$, $\bigwedge\limits_{\ell<n} [G^\ell \cap N_0 = G^*]$ where $G^* \subseteq P_i \cap N_0$ is generic over N_0 and $Y =: \bigcap\limits_{\ell<n} \mathscr{M}[G^\ell, N_1] \in D_\alpha(N)$ and $p \in P_j \cap N_0, p \restriction i \in G^*$. Then for some $G^{**} \subseteq P_j \cap N_0$ generic over N_0 we have $p \in G^{**} \in N_1$ and $\bigwedge\limits_{\ell<n} \bigvee\limits_{q \in G^\ell} [q \Vdash$ "G^{**} has an upper bound in $P_j/\underset{\sim}{G}_{P_i}$"$]$.

Remark. We may like to phrase clause (c) as a condition on each $\underset{\sim}{Q}_i$, for this see Definition 2.6, 2.8, 2.14; this is a slight loss if we deal with the case $i < j, i$ non limit $\Rightarrow P_j/P_i$ is proper. As no need arise here we ignore this.

1.9 Main Claim. *Assume $\bar{Q}$ is a CS iteration, $\bar{Q} \in \mathscr{H}(\chi_0^{\mathfrak{p}})$ and $\mathscr{P}(Lim\, \bar{Q}) \in \mathscr{H}(\chi_0^{\mathfrak{p}}))$, $\mathfrak{p}$ a reasonable parameter of length ω_1, $\delta = \ell g(\bar{Q})$ is a limit ordinal and for every $\alpha < \delta, \bar{Q} \restriction \alpha$ is a $\mathfrak{p} - NNR^0_{\aleph_0}$ iteration and $\mathfrak{p}$ is a $\eth$-winner.*

Then $\bar{Q}$ is a $\mathfrak{p} - NNR^0_{\aleph_0}$ iteration.

Proof.

Proof of clause (a) of Definition 1.8:

Trivial.

Proof of clause (b) of Definition 1.8:

Follows from clause (d) of Definition 1.8 proved below.

Proof of clause (d):

Let $i, j, \alpha, N_0, N_1, n, q_0, \ldots, q_{n-1}, G^\ell, G^*, p$ be as in the assumptions of clause (d). Let $\alpha' =$ $\text{otp}(N_0 \cap [i,j))$, and $\alpha' < \omega_1$ so $\alpha' \in N_1$. If $j < j(*)$ use "$\bar{Q} \restriction j$ is a $\mathfrak{p} - NNR^0_{\aleph_0}$ iteration", so assume $j = j(*)$; if $i = j$ the conclusion is trivial so assume $i < j$. Let $i_m \in N_0 \cap j$ be such that $i_0 = i, i_m < i_{m+1}$ and $\langle i_m : m < \omega \rangle \in N_1$ and

[2] so naturally $\ell g(\mathfrak{p}) = \omega_1$

$\bigcup_{m<\omega} i_m = \sup(N_0 \cap j)$. Choose M_ℓ for $\ell < 5$ such that $y^* =: \{i, j, \alpha, \alpha', \bar{Q}, N_0, \langle i_m : m < \omega\rangle\} \in M_\ell \in \mathscr{E}_{\alpha'} \cap N_1 \cap \bigcap_{\ell<n} \mathscr{M}[G^\ell, N_1], M_0 \in M_1 \in M_2 \in M_3 \in M_4$ and $\bigcap_{\ell<n} \mathscr{M}[G^\ell, M_0, y^*] \in D_\alpha(M_0)$.

Now for $\ell < n$ we can choose $q'_\ell \in G^\ell \cap M_4$ which forces (for $P_{i_0} = P_i$) a value to $\underset{\sim}{G}_{P_{i_0}} \cap M_3$ which necessarily is $G^\ell \cap M_3$ and necessarily $q'_\ell \leq q_\ell$ and, of course, q'_ℓ is (M_k, P_{i_0})-generic forcing $\underset{\sim}{G}_{P_{i_0}} \cap M_k = G^\ell \cap M_k$ for $k = 0, 1, 2, 3$ and $\underset{\sim}{G}_{P_{i_0}} \cap N_0 = G^*$.

Let $\langle \mathscr{I}^*_m : m < \omega\rangle \in M_0$ list the maximal antichains of P_j that belongs to N_0. Now we choose by induction on $m < \omega$, the objects $r_m, G^*_m, p_m, n_m, G^\ell_m$ (for $\ell < n_m$) and y_m such that:

- (a) $r_m \in P_{i_m} \cap M_4$
- (b) $\mathrm{Dom}(r_m) \subseteq [i, i_m)$
- (c) $r_{m+1} \restriction i_m = r_m$
- (d) $q'_\ell \cup r_m (\in P_{i_m})$ is (M_k, P_{i_m})-generic for $k = 0, 1, 2, 3$ and is (N_0, P_{i_m})-generic (note that q'_ℓ, r_m have disjoint domains)
- (e) for every predense subset $\mathscr{I}$ of P_{i_m} which belongs to M_2, for some finite $\mathscr{J} \subseteq \mathscr{I} \cap M_2$ the set $\mathscr{J}$ is predense above $q'_\ell \cup r_m$ for each $\ell < n$
- (f) $n_m < \omega$ and for $\ell < n_m$ we have: G^ℓ_m is a subset of $P_{i_m} \cap M_0$ generic over $M_0, G^\ell_m \in M_1$
- (g) if $\ell < n_{m+1}$ then $G^\ell_{m+1} \cap P_{i_m} \in \{G^k_m : k < n_m\}$
- (h) $n_0 = n, G^\ell_0 = G^\ell \cap M_0$
- (i) $q'_\ell \cup r_m \Vdash_{P_{i_m}}$ "$\underset{\sim}{G}_{P_{i_m}} \cap M_0 \in \{G^\ell_m : \ell < n_m\}$"
- (j) G^*_m is a subset of $P_{i_m} \cap N_0$ generic over N_0
- (k) $G^*_m \subseteq G^\ell_m$ for $\ell < n_m$
- (l) $p_m \in P_j \cap N_0, p_m \restriction i_m \in G^*_m, p_{m+1} \in \mathscr{I}^*_m, p_0 = p, p_m \leq p_{m+1}$
- (m) $Y_m =: \bigcap_{\ell<n_m} \mathscr{M}[G^\ell_m, M_0, y^*] \in D_{\alpha'}(M_0)$ where $y^* =: \{N_0, \langle i_k : k < \omega\rangle, \bar{Q}, i, j\}$.

<u>Why is this sufficient?</u>

During the construction above we choose inductively members of M_4 and all the parameters used are from M_4, so if we choose a well ordering $<^*$ of M_4 and always choose the $<^*$-first object the construction is determined. Clearly there is such $<^* \in N_1$. Now

(α) $r = \bigcup_m r_m$ (i.e. the unique $r \in P_j$ satisfying $m < \omega \Rightarrow r \restriction i_m = r_m$) belongs to P_j and to N_1,

(β) $G^{**} = \{p' \in P_j \cap N_0 : \bigvee_{m<\omega} [p' \le p_m]\}$ belongs to M_4 and is a subset of $P_j \cap N_0$ generic over N_0,
(by the choice of $\langle \mathscr{I}^*_m : m < \omega \rangle$ and clause (k)) and $q'_\ell \cup r$ is above G^{**} (in P_j).

So we are done.

Why can we carry out the construction?

For $m = 0$ there is no problem. So assume we have it for m and we shall choose for $m + 1$.

Stage A: Choose $p_{m+1} \in N_0 \cap \mathscr{I}^*_m$ such that $p_m \le p_{m+1}$ and $p_{m+1} \restriction i_m \in G^*_m$. No problem.

Stage B: Choose $G^*_{m+1} \subseteq P_{i_{m+1}} \cap N_0$ generic over $N_0, G^*_m \subseteq G^*_{m+1} \in M_0, p_{m+1} \restriction i_{m+1} \in G^*_{m+1}$ and $\bigwedge_{\ell<n_m} \bigvee_{r \in G^\ell_m} [r \Vdash_{P_{i_m}}$ "G^*_{m+1} has an upper bound in $P_{i_{m+1}}/\underset{\sim}{G}_{P_{i_m}}$].

This is easy by "$\bar{Q} \restriction i_{m+1}$ is a $\mathfrak{p}-NNR^0_{\aleph_0}$ iteration" applied with $i_m, i_{m+1}, \alpha', p_{m+1} \restriction i_m, G^*_m, \langle G^\ell_m : \ell < n_m \rangle, N_0, M_0$ here standing for $i, j, \alpha, p, G^*, \langle G^\ell : \ell < n \rangle, N_0, N_1$ there (i.e. we use clause (d) of the Definition 1.8); we are using $\mathrm{otp}(N_0 \cap [i_m, i_{m+1})) < \mathrm{otp}(N_0 \cap [i_m, j)) = \alpha'$.

Stage C:

Now $\underset{\sim}{G}_{P_{i_m}} \cap M_1$ is a P_{i_m}-name of an object from V (as P_{i_m} is proper not adding reals), so $\mathscr{I} =: \{p \in P_{i_m} : p$ forces a value to $\underset{\sim}{G}_{P_{i_m}} \cap M_1 (\in V)\}$ is a dense open subset of P_{i_m} and $\mathscr{I} \in M_2$. By clause (e) in the induction hypothesis there is a finite $\mathscr{J} \subseteq \mathscr{I} \cap M_2$ such that: $\ell < n \Rightarrow \mathscr{J}$ is predense above $q'_\ell \cup r_m$. Without loss of generality $\mathscr{J}$ is minimal. Let $n_{m+1} = |\mathscr{J}|$.
Let $\mathscr{J} = \{p^\ell_m : \ell < n_{m+1}\}$ let $p^\ell_m \Vdash$ "$\underset{\sim}{G}_{P_{i_m}} \cap M_1 = H^\ell_m$". So $H^\ell_m \in M_2$, and as $\mathscr{J}$ is minimal, $H^\ell_m \cap M_0 \in \{G^\ell_m : \ell < n_m\}$ so let $H^\ell_m \cap M_0 = G^{h(\ell)}_m$ where $h : n_{m+1} \to n_m$.

Let $Y =: \bigcap_{\ell < m_n} \mathscr{M}[G^\ell_m, M_0, y^*] \in D_{\alpha'}(M_0)$.

Now we choose by induction on $\ell \le n_{m+1}$ a condition $r^\ell_m \in M_1$ such that:

(α) $r^\ell_m \in P_{i_{m+1}} \cap M_1, r^\ell_m \restriction i_m \in H^\ell_m$

(β) r^ℓ_m is (M_0, P_{i_m})-generic and force a value to $\underset{\sim}{G}_{P_{i_m}} \cap M_0$ called G^ℓ_{m+1}

(γ) r^ℓ_m is above G^*_{m+1} (chosen in the previous stage), moreover above $p^{h(\ell)}_m$

(δ) $Y \cap \bigcap\limits_{k<\ell} \mathscr{M}[G^k_{m+1}, M_0, y^*] \in D_{\alpha'}(M_0)$.

For the induction step, apply clause (c) in Definition 1.8 (as $\bar{Q} \restriction i_{m+1}$ is a $\mathfrak{p} - NNR^0_{\aleph_0}$-iteration) with $i_m, i_{m+1}, \alpha', M_0$, large enough member of $H^\ell_m, p^{h(\ell)}_m$, $Y^\ell_m = Y \cap \bigcap\limits_{k<\ell} \mathscr{M}[G^k_{m+1}, M_0, y^*]$ here standing for i, j, α, N, q, p, Y there, noting $Y^\ell_m \in M_1$.

Stage D:

We can choose r_{m+1} as required such that $\{r^\ell_m : \ell < n_{m+1}\}$ is predense over it by [Sh:f, Ch.XVIII,2.6] (can first do it for each r^ℓ_m separately and then put them together).

So we have finished proving clause (d).

Proof of clause (c).

We prove this by induction on α.

Let i, j, α, N, p, q, Y be as there. If $j < j(*)$ we can apply "$Q \restriction j$ is a $\mathfrak{p} - NNR^0_{\aleph_0}$ iteration", so without loss of generality $j = j(*)$. If $i = j$ the statement is trivial so assume $i < j$. Choose i_n for $n < \omega$ such that $i_0 = i, i_n \in N \cap j, i_n < i_{n+1}$ and $\bigcup\limits_{n<\omega} i_n = \sup(j \cap N)$. Let $\langle (y_n, \beta_n) : n < \omega \rangle$ list the pairs $(y, \beta) \in N \times (\alpha \cap N)$ such that $y \in \mathscr{H}(\chi^{\mathfrak{p}}_\beta)$. Let **St** be a winning strategy for the chooser in the game $\Game_\alpha(N)$. Let $\langle \mathscr{I}_n : n < \omega \rangle$ list the dense open subsets of P_j which belongs to N.

Now we choose by induction on $n < \omega$, the objects $q_n, \underset{\sim}{p}_m, \underset{\sim}{M}_n, \underset{\sim}{Y}_m$ such that:

(a) $q_n \in P_{i_n}, q_0 = q$

(b) q_n is (N, P_{i_n})-generic

(c) $q_{n+1} \restriction i_n = q_n$

(d) $\underset{\sim}{p}_n$ is a P_{i_n}-name of a member of $(P_j / \underset{\sim}{G}_{P_{i_n}}) \cap N$

(e) $\underset{\sim}{p}_n$ is forced to belong to $\mathscr{I}_n$

(f) $\underset{\sim}{M}_n$ is a P_{i_n}-name of a member of $\mathscr{E}_{\beta_n} \cap N$

(g) if $G_j \subseteq P_j$ is generic over $V, q_n \in G_j, \underset{\sim}{p}_n[G_j \cap P_{i_n}] \in G_j, M = \underset{\sim}{M}_n[G_j]$, <u>then</u>

- (α) $G_j \cap M$ is a subset of $Q \cap M$ generic over M
- (β) $\mathscr{M}[G_j \cap M, M, y^*] \cap Y \in D_{\beta_n}[M]$
- (γ) $\underset{\sim}{p}_n[G_j]$ belongs to M

(h) $\langle Y_m \cap \mathscr{M}[\underset{\sim}{G}_{i_m}, \underset{\sim}{M}_m, y^*], \underset{\sim}{Y}_m, P_{i_m}, \underset{\sim}{M}_m : m \leq n\rangle$ is forced by q_n to be an initial segment of a play of the game $\eth_\alpha(N)$ in which the chooser uses the fixed winning strategy **St**.

The proof is straight by the induction hypothesis on β and "$\bar{Q} \restriction i_n$ is a $\mathfrak{p} - NNR^0_{\aleph_0}$-iteration" remembering that $\underset{\sim}{M}_n, \underset{\sim}{Y}_n$ are P_{i_n}-names but of objects in V.
Alternatively, see more in the proof in §4 or see below. $\square_{1.9}$

1.10 Remark. 1) We could have used the "adding no reals" and clause (d) in the proof of clause (c) in order to weaken "winner" to "not loser"; also we can use $\eth'_\alpha(N, N', P)$, see §4. Also the "$N_0 \in \bigcup\limits_{\beta<\alpha} \mathscr{E}_\beta$" can be replaced by "$N_0 \in \mathscr{E}'_0$" with $\mathscr{E}'_0 \subseteq [\mathscr{H}(\chi^{\mathfrak{p}}_0)]^{\aleph_0}$ stationary; also we can put extra restrictions on G^* (and G^{**}), e.g. $\mathscr{M}[G^*, N_0, y^*]$ large.
2) Of course, the use of $\langle \chi_\alpha : \alpha < \ell g(\mathfrak{p})\rangle$ is not really necessary, we could have used subsets of $\mathscr{P}(P_{\ell g(\bar{Q})})$, (so $\mathscr{E}_\alpha$ is changed accordingly) as all the properties depend just on $N_\ell \cap \mathscr{P}(P_{\ell g(\bar{Q})})$. We feel the present way is more transparent.
3) The proof of clause (c) being preserved can be applied to $\bar{Q}$ satisfying (a) + (c) of Definition 1.8 (so possibly adding reals), but then we have to replace $D_\alpha(N)$ by a definition of such pseudo filters with the winning strategy being absolute enough, e.g. for standard $\bar{D}$.
4) We can also replace $D_\alpha(a)$ by a partial ordered set $L_\alpha(a)$ and a function v_a : $L_\alpha(a) \rightarrow \mathscr{P}(a)$ (i.e. by (L_a, v_a)).
5) In the case we adopt remark (3), then remark (1) (non-losing) becomes less clear, as the universe changes during the proof. We may consider to weaken "the chooser has a winning strategy" in $\eth'_\alpha(N)$ (game depends on $\bar{D}$), e.g. to not losing in a game with finitely many boards, possibly splitting in each move (no real novelty in the proof), but it is not clear how interesting this is. But if $D_\alpha(N)$ is inductively defined as sums over an ultrafilter (which is preserved), it just seems that not losing is enough.

1.11 Definition. $\bar{Q}$ a CS iteration will be called $\mathfrak{p}$-proper if clauses (a) + (c) of Definition 1.8 holds.

<u>1.12 Question</u>: Is this notion of interest in iterations adding reals?

1.13 Definition. Let $\kappa \in [2, \omega]$. We say that $\bar{Q}$ is a $\mathfrak{p} - NNR^0_\kappa$-iteration if: from Definition 1.8 we have:

- (a) as there
- (b) as there
- (c) [long properness] as there
- (d) [κ-anti w.d.] as in (d) there but $1 + n < \kappa$ and $N_0 \in \mathscr{E}_0$.

1.14 Main Claim. *Assume $2 \leq \kappa < \omega, \bar{Q}$ is a CS iteration, $\mathscr{P}(Lim(\bar{Q})) \subseteq \mathscr{H}(\chi_0^{\mathfrak{p}}), \mathfrak{p}$ a reasonable parameter of length ω_1 which is a non-$\eth'$-loser, $\delta = \ell g(\bar{Q})$ a limit ordinal and $i < \delta \Rightarrow \bar{Q} \restriction i$ is a $\mathfrak{p} - NNR_\kappa$-iteration. <u>Then</u> $\bar{Q}$ is a $\mathfrak{p} - NNR^0_\kappa$-iteration.*

Remark. 1) As it is just combining the proof of 1.9 and [Sh:f, Ch.XVIII,§2] (or [Sh:b, Ch.VIII,4.10]-[Sh:f, Ch.VIII,4.10]) we elaborate less.
2) See more in §4.

Proof. Similar to the proof of 1.9, with some changes (as in [Sh:b, Ch.VIII,§4], [Sh:f, Ch.VIII,§4], [Sh:f, Ch.XVIII,2.10C]). During the proof of clause (d) we add to clauses (a)-(m):

- (n) n_m is a power of 2, say $2^{n^*_m}$ and so we can rename $\{G^\ell_m : \ell < n_m\}$ as $\{G^\eta_m : \eta \in {}^{n^*_m}2\}$
- (o)
  - (α) $M_\eta \in M_1 \cap \mathscr{E}_{\beta_\eta}$ for $\eta \in {}^{(n^*_m \geq)}2$ where $j_0 =$ otp$([i,j) \cap N_0)$ and if $\eta = \nu\hat{\ } < i >$ then $j_\eta =$ otp$([i,j) \cap M_\nu)$
  - (β) $M_{<>} = N_0$
  - (γ) $M_\eta \in M_{\eta\hat{\ }\langle 0\rangle} \cap M_{\eta\hat{\ }\langle 1\rangle}$
  - (δ) $\eta \triangleleft \nu_1 \in {}^{n^*_m}2 \ \ \& \ \ \eta \triangleleft \nu_2 \in {}^{n^*_m}2 \Rightarrow G^{\nu_1}_m \cap M_\eta = G^{\nu_2}_m \cap M_\eta$ so we call it K^η_m
  - (ε) $M_{\eta\hat{\ }\langle 0\rangle} = M_{\eta\hat{\ }\langle 1\rangle}$ when $\eta \in {}^{n^*_m}2$ call it N_η
  - (ζ) $N_\eta \in \mathscr{E}_{\ell g(\eta)}$ for $\eta \in {}^{(m^*_m >)}2$
  - (η) $Y^\eta_m = \mathscr{M}[K^{\eta\hat{\ }\langle 0\rangle}_m, N_\eta] \cap \mathscr{M}[K^{\eta\hat{\ }\langle 1\rangle}_m, N_\eta] \in D_{j_{\eta\hat{\ }<0>}}(N_\eta)$.

$\square_{1.14}$

1.15 Conclusion. Let $\bar{Q}$ be a CS iteration and $\mathfrak{p}$ a non-$\eth'$-loser reasonable parameter, $2 \le n(*) \le \aleph_0, \ell g(\mathfrak{p}) = \omega_1$.
Then $\bar{Q}$ is a $\mathfrak{p} - NNR^0_{n(*)}$-iteration iff for each $i < \ell g(\bar{Q})$

- $(*)_i$ $\underset{\sim}{Q}_i$ is a proper forcing and $P_i, \underset{\sim}{Q}_i$ satisfies clauses (d) + (c) of the Definition "$\mathfrak{p} - NNR^0_{n(*)}$-iteration" with $i, i+1$ here standing for i, j there.

Proof. By induction on $j = \ell g(\bar{Q})$. For $j = 0$ there is nothing, for j limit use 1.9 (or 1.14) and if j successor just read the definitions. $\square_{1.15}$

We point out here that clause (c) of Definitions 1.8 and 1.13 really follows from earlier properties which play parallel roles.

1.16 Claim. *1) Assume that $\mathfrak{p}$ is a standard reasonable parameter, and is standard and $\alpha < \ell g(\mathfrak{p}), N \in \mathscr{E}^{\mathfrak{p}}_\alpha, Y \in D_\alpha(N)$ and $\delta \le \omega_1 \cap N$ a limit ordinal.*
Then we can find sequences $\bar{N} = \langle N_i : i < \delta\rangle, \langle \gamma_i : i < \delta\rangle$ such that:

- (a) *$N_i \in N$ (for $i < \delta$) is countable, $N \cap \alpha \subseteq N_i$ and $N_i \in Y$*
- (b) *$N_i \subseteq \bigcup\limits_{\beta \in \alpha \cap N} (\mathscr{H}(\chi^{\mathfrak{p}}_\beta), \in)$, and $\beta \in \alpha \cap N_i \Rightarrow N_i \restriction \mathscr{H}(\chi^{\mathfrak{p}}_\beta) \prec (\mathscr{H}(\chi^{\mathfrak{p}}_\beta), \in)$*
- (c) *$i < j \Rightarrow N_i \subseteq N_j$*
- (d) *for i limit $N_i = \bigcup\limits_{j<i} N_j$ and $N \cap \bigcup\{\mathscr{H}(\chi^{\mathfrak{p}}_\beta) : \beta \in \alpha \cap N\} = \bigcup\limits_{j<\delta} N_j$ so we can stipulate $N_\delta = N$*
- (e) *$\beta \in \alpha \cap N \Rightarrow \langle \mathscr{H}(\chi^{\mathfrak{p}}_\beta) \cap N_j : j \le i\rangle \in N_{i+1}$*
- (f) *$\gamma_i \in N_i \cap \alpha, N_i \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\gamma_i}) \in \mathscr{E}_{\gamma_i} \cap Y, \bar{\gamma} \restriction (i+1) \in N_{i+1}$*
- (g) *if $i \le \delta$ is a limit ordinal, $(i = \delta \ \& \ \beta \in \alpha \cap N_i) \vee (i < \delta \ \& \ \beta \in \gamma_i \cap N_i)$ (so $\beta \in \mathscr{H}(\chi^{\mathfrak{p}}_\beta)$) and $y \in \mathscr{H}(\chi^{\mathfrak{p}}_\beta)$, then for some $j < i, \gamma_j = \beta, y \in N_j$. Moreover*
- $(g)^+$ *if $i \le \delta$ is a limit ordinal, then $\{N_j \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\gamma_j}) : j < i$ and $\gamma_j < \gamma_i\} \in D_{\gamma_i}(N) = D_{\gamma_i}(N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\gamma_i}))$*
- (h) *if $\delta < N \cap \omega_1$ then $\delta \in N_0$, if $\delta = N \cap \omega_1$ then $i < \delta \Rightarrow i \in N_i$.*

2) If $\mathfrak{p}$ is a standard, reasonable parameter, $\bar{Q}$ is a CS iteration, $\ell g(\bar{Q}) = \alpha+1, \bar{Q} \restriction \alpha$ is $\mathfrak{p} - NNR^0_{k(*)}$-iteration and

$\Vdash_{P_\alpha}$ "$\underset{\sim}{Q}_\alpha$ *is proper and* $(<^+ \omega_1)$*-proper", then trying to apply 1.14 for* $\bar{Q}$*, clauses (a),(c) (called* $\mathfrak{p}$*-proper in Definition 2.3 below) holds.*
3) If $\ell g(\mathfrak{p}) = \omega_1$*, then in part (2) it suffices to ask* $\Vdash_{P_i}$ "$\alpha < \omega_1 \Rightarrow \underset{\sim}{Q}_i$ *is* α*-proper", that is* $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ *is* $(< \omega_1)$*-proper".*

Proof. 1) By induction on $\beta \in (\alpha \cap N) \cup \{\alpha\}$ we prove that there is $\langle N_j : j < \beta \cap N\rangle \in N$ satisfying the relevant requirements.
2) By Definition of $(< \omega_1)$-proper, if $G_\alpha \subseteq P_\alpha$ is generic over $\alpha, Y =: \mathscr{M}[G_\alpha, N, y^*] \in D_\alpha(N)$, let $\langle N_i : i < \delta\rangle$ be as in (1) for $\delta = N \cap \omega_1$, without loss of generality $p \in N_0 \cap \underset{\sim}{Q}_\alpha[G_\alpha]$, let $q \geq p$ be $(N_i, \underset{\sim}{Q}_\alpha[G_\alpha])$-generic for every $i < \delta$ (formally look at $\bar{N}' = \langle N'_i : i < \delta\rangle, N'_i = N_i \restriction \mathscr{H}(\chi^{\mathfrak{p}}_0)$ and apply to it the $(< \omega_1)$-proper).
3) Follows as $\alpha \in N \Rightarrow \delta = \omega\alpha \in N \cap \omega_1$. $\square_{1.16}$

<u>1.17 Discussion</u>: 1) This includes as special cases [Sh:b, Ch.V,§5,§7]. There is no direct comparison with [Sh:b, Ch.VIII,§4], [Sh:f, Ch.VIII,§4], but we can make the notion somewhat more complicated, to include the theorem there in our context, i.e., what is not included is a generalization there not really needed for the examples discussed there (see here in §3 below). The condition there involves having many sequences $\langle N_\alpha : \alpha \leq \delta\rangle$ such that if $p_0, p_1 \in P, p_\ell$ is (N_i, P_α)-generic for $i, p_\ell \Vdash$ "$\underset{\sim}{G}_P \cap N_0 = G^*$", then there is $G' \subseteq \text{Gen}(\underset{\sim}{Q}_i[G^*], N_0[G^*]), G' \in N_0, P_i \nVdash_{P_i}$ "$\underset{\sim}{G}_0$ has no bound in $\underset{\sim}{Q}_\alpha$". This speaks on a family of sequences from $[\mathscr{H}(\chi)]^{\aleph_0}$ rather than members.
2) For [Sh:f, Ch.XVIII,§2], the comparison is not so easy. Our problem is to "carry" good $(N, P_i, \langle G_\ell : \ell < n\rangle), G_\ell \in \text{Gen}(N, P_i)$ with a bound, such that we can "increase i" and we can find $N', y \in N' \in N, N' \prec N$ such that $(N', P_i, \langle G_i \cap N : \ell < n\rangle)$ is good enough. In [Sh:f, Ch.XVIII] we are carrying genericity in some $P_{\bar{\alpha}}, \bar{\alpha} \in \text{trind}(i)$, here much less. But what we need is the implication "if $(N, P_i, \bar{G})$ is good we can extend it to good $(N, P_{i+1}, \bar{G}')$", so making good weaker generates an incomparable notion and clearly there are other variants. We can consider other such notions (see §5).
2A) We can give alternative proofs of consistency of the questions in [Sh:f, Ch.XVIII,§1] by the present iteration theorem (see §3 below).
3) We can in 1.16(2) weaken the assumption "$(< \omega_1)$-proper" to things of the form: if $\bar{N} = \langle N_i \leq \delta\rangle, p \in Q \cap N_0, Q \in N_0$ then there is $q \geq p$ such that $q \Vdash$ "for many $i < \delta, \underset{\sim}{G}_Q \cap N_i \in \text{Gen}(N_i, Q)$". In particular the condition applies to the forcing notions considered in [Sh:f, Ch.XVIII,§1].

§2 Delayed properness

In this section we prove little, but the notions introduced are used in §3, §4. We concentrate here on simple parameters so the reader may assume it all the time. We give two versions, the simpler one is version 2 for which simplicity is a very natural demand.

2.1 Observation. 1) Assume

- (a) $\bar{\chi} = \langle \chi_\alpha : \alpha < \alpha^* \rangle$ increases fast enough, so $\bigcup\limits_{\beta<\alpha} \mathscr{H}(\chi_\beta) \in \mathscr{H}(\chi_\alpha)$
- (b) $\mathscr{E}_\alpha \subseteq \{N : N$ a countable elementary submodel of $(\mathscr{H}(\chi_\alpha), \in)\}$ is stationary
- (c) $R_\alpha \in \mathscr{H}(\chi_\alpha)$ and $N \in \mathscr{E}_\alpha$ implies $\langle \chi_\beta : \beta < \alpha \rangle \in N, \langle R_\beta : \beta \le \alpha \rangle \in N$ and $\langle \mathscr{E}_\beta : \beta < \alpha \rangle \in N$.

Then there is a standard reasonable parameter $\mathfrak{p}$ with $\ell g(\mathfrak{p}) = \alpha^*, \mathscr{E}^{\mathfrak{p}}_\alpha = \mathscr{E}_\alpha$, $R^{\mathfrak{p}}_\alpha = R_\alpha$ (and $\bar{D}^{\mathfrak{p}}$ standard).
2) If in addition clause (d) below hold, then $\mathfrak{p}$ is a simple standard reasonable parameter where

- (d) $\beta \in N \in \mathscr{E}_\alpha, \beta < \alpha \Rightarrow N \cap \mathscr{H}(\chi_\beta) \in \mathscr{E}_\beta$.

3) If $\chi_\alpha = (\beth_{2\alpha+1})^+$ for $\alpha < \alpha^*, R_\alpha \in \mathscr{H}(\chi_\alpha)$, then χ_α increases fast enough. If $\langle \chi_\alpha : \alpha < \alpha^* \rangle, \langle R_\alpha : \alpha < \alpha^* \rangle$ are as in part (1), $\chi \le \chi_0, \mathscr{E} \subseteq [\mathscr{H}(\chi)]^{\le \aleph_0}$ stationary and we let $\mathscr{E}_\alpha = \{N : N$ a countable elementary submodel of $(\mathscr{H}(\chi_\alpha), \in)$ and $\langle \chi_\beta : \beta < \alpha \rangle, \langle R_\beta : \beta \le \alpha \rangle, \mathscr{E}$ belong to N and $N \cap \mathscr{H}(\chi) \in \mathscr{E}\}$, then the assumptions of parts (1) and (2) above holds (hence their conclusions).

Proof. Straight.

2.2 Definition. Let $\mathfrak{p}$ be a reasonable parameter and $\alpha \le \beta < \ell g(\mathfrak{p})$.
1) For $N \in \mathscr{E}_\beta$ such that $\alpha \in N$ we define a game $\eth_{\alpha,\beta}(N) = \eth_{\alpha,\beta}(N, \mathfrak{p})$ as follows. A play lasts ω-moves. In the n-th move:

- (a) the challenger chooses $X_n \in D_\beta(N)$ such that $m < n \Rightarrow X_n \subseteq X_m$
- (b) the chooser chooses $\alpha_n \in \alpha \cap N$
- (c) the challenger chooses $\beta'_n \in \beta \cap N$ and $y'_n \in N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_n})$

(d) the $\underline{\text{chooser}}$ chooses $\beta_n \in \beta \cap N \backslash \beta'_n$ and $M_n \in X_n \cap \mathscr{E}_{\beta_n}$ and $y_n \in M_n \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_n})$ satisfying $\alpha_n \le \beta_n, y'_n \in M_n, \alpha_n \in M_n$ and $Y_n \in D_{\beta_n}(M_n)$ such that $Y_n \subseteq X_n$ and $Y_n \in N$

(e) the $\underline{\text{challenger}}$ chooses $M'_n \in Y_n \cap \mathscr{E}_{\alpha_n} \cap (M_n \cup \{M_n \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_n})\})$ satisfying $y_n, y'_n \in M'_n$ and chooses $Z_n \in D_{\alpha_n}(M'_n) = D_{\alpha_n}(M'_n \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_n}))$ such that $Z_n \subseteq Y_n$.

In the end the chooser wins the play if

$$\bigcup \{Z_n \cup \{M'_n\} : n < \omega\} \in D^+_\alpha(N)\,(= D^+_\alpha(N \cap \mathscr{H}(\chi^{\mathfrak{p}}_\alpha)))$$

1A) We call $\Game_{\alpha,\beta}(N) = \Game_{\alpha,\beta}(N,\mathfrak{p})$ version 1; version 2 means that in clause (e) we add the requirement $M'_n = M_n \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_n})$ and in clause (d) we require $\alpha_n \le_{\mathfrak{p}} \beta_n$. If we do not mention the version it means that it holds for both.
2) Assume $N \in N' \prec (\mathscr{H}(\chi), \in)$. We define a game $\Game'_{\alpha,\beta}(N, N', \mathfrak{p})$ similarly but replace a) - e) by

$(a)'$ the $\underline{\text{challenger}}$ chooses $X_n \in D_\beta(N)$ such that $m < n \Rightarrow X_n \subseteq X'_m$

$(b)'$ the $\underline{\text{chooser}}$ chooses $\alpha_n \in \alpha \cap N$ and $X'_n \subseteq X_n$ such that $X'_n \in D_\beta(N)$

$(c)'$ like (c)

$(d)'$ like (d) but we replace "$Y_n \in N$" by "$Y_n \in N'$"

$(e)'$ like (e) but add $Z_n \in N'$.

Note: so every proper initial segment of a play belongs to N'.

2.3 Definition. Let $\mathfrak{p}$ be a reasonable parameter.
1) For $\alpha \le \beta < \ell g(\mathfrak{p})$ we say $\mathfrak{p}$ is an $\Game_{\alpha,\beta}$-winner [non-$\Game_{\alpha,\beta}$-loser] when for some $x \in \mathscr{H}(\chi^{\mathfrak{p}}_\beta)$ we have:

> if $\{x, \alpha\} \in N \in \mathscr{E}_\beta$, $\underline{\text{then}}$ the chooser wins the game $\Game_{\alpha,\beta}(N,\mathfrak{p})$ [the challenger does not win in the game $\Game_{\alpha,\beta}(N,\mathfrak{p})$].

2) We can replace $\Game_{\alpha,\beta}$ by $\Game'_{\alpha,\beta}$.
3) For any function $f : \ell g(\mathfrak{p}) \to \mathscr{P}(\ell g(\mathfrak{p}))$ we can replace α, β by f meaning: for every $\alpha < \ell g(\mathfrak{p})$ and $\beta \in f(\alpha)$, we have $\mathfrak{p}$ is a $\Game_{\alpha,\beta}$-winner.
4) Let $\mathscr{F}^{\mathfrak{p}}$ be the family of functions from $\ell g(\mathfrak{p})$ to $\mathscr{P}(\ell g(\mathfrak{p}))$ such that for each $\alpha < \ell g(\mathfrak{p}), f(\alpha)$ is a nonempty subset of $[\alpha, \ell g(\mathfrak{p}))$. Let $\mathscr{F}^{\mathfrak{p}}_{\text{club}}$ be the set of $f \in \mathscr{F}^{\mathfrak{p}}$ such that for each $\alpha < \ell g(\mathfrak{p}), f(\alpha)$ is a club of $\ell g(\mathfrak{p})$. Let $\mathscr{F}^{\mathfrak{p}}_{\text{end}}$ be the set of $f \in \mathscr{F}^{\mathfrak{p}}$ such that for each $\alpha < \ell g(\mathfrak{p}), f(\alpha)$ is an end segment of $\ell g(\mathfrak{p})$, we then may identify $f(\alpha)$ with $\text{Min}(f(\alpha))$.

5) We call f decreasing continuous if $\alpha_1 < \alpha_2 < \ell g(\mathfrak{p}) \Rightarrow f(\alpha_2) \subseteq f(\alpha_1)$ and for limit $\alpha < \ell g(\mathfrak{p})$ we have $f(\alpha) = \bigcap\limits_{\gamma<\alpha} f(\gamma)$. Let $f \le g$ mean $(\forall \alpha < \ell g(\mathfrak{p}))(g(\alpha) \subseteq f(\alpha))$.

There are obvious monotonicity properties.

2.4 Claim. *Assume $\mathfrak{p}$ is a reasonable parameter.*
1) If $\alpha \le_{\mathfrak{p}} \alpha', \alpha' \le \alpha \le \beta = \beta' < \ell g(\mathfrak{p})$, and $\mathfrak{p}$ is a $\Game_{\alpha,\beta}$-winner, <u>then</u> it is $\Game_{\alpha',\beta'}$-winner. Similarly for $\Game'$-winner, non-$\Game$-loser, non-$\Game'$-loser.
2) If $\mathfrak{p}$ is a $\Game_{\alpha,\beta}$-winner, <u>then</u> $\mathfrak{p}$ is a $\Game'_{\alpha,\beta}$-winner and a non-$\Game_{\alpha,\beta}$-loser. If $\mathfrak{p}$ is a $\Game'_{\alpha,\beta}$-winner or non-$\Game_{\alpha,\beta}$-loser, <u>then</u> $\mathfrak{p}$ is non-$\Game'_{\alpha,\beta}$-loser.
3) Assume $f, g \in \mathscr{F}^{\mathfrak{p}}$ and $f \le g$. If $\mathfrak{p}$ is a $\Game_f$-winner [or $\Game'_f$-winner] [or non-$\Game_f$-loser] [or non-$\Game_f$-loser], <u>then</u> $\mathfrak{p}$ is a $\Game_g$-winner [or $\Game'_g$-winner] [or non-$\Game_g$-loser] [or non-$\Game'_g$-loser].

Proof. Straight (we are using the simplicity of $\mathfrak{p}$).

2.5 Claim. *1) Assume $\mathfrak{p}$ is a standard reasonable parameter. <u>Then</u> $\mathfrak{p}$ is a winner (see Definition 1.5).*
2) If $\mathfrak{p}$ is a reasonable parameter and it is a winner, <u>then</u> $\mathfrak{p}$ is a $\Game_{\alpha,\alpha}$-winner.
3) If $\mathfrak{p}$ is a reasonable parameter and it is a winner and $\alpha \le \beta < \ell g(\mathfrak{p})$ <u>then</u> $\mathfrak{p}$ is a $\Game_{\alpha,\beta}$-winner (hence $\Game_f$-winner for f as in 2.3(3)).
4) Similarly with $\Game'$-winner, $\Game'_{\alpha,\beta}$ winner and/or with the "non loser" cases.

Proof. Straight.

2.6 Definition. 1) Let $\mathfrak{p}$ be a reasonable parameter and P be a proper forcing notion not adding reals, $\mathscr{P}(P) \in \mathscr{H}(\chi_0^{\mathfrak{p}})$ and $G_P \subseteq P$ is generic over V.

<u>Then</u> we interpret $\mathfrak{p}$ in V^P as $\mathfrak{p}' = \mathfrak{p}^{V[G]}$, or we may write $\mathfrak{p}^P$ defined as follows:

(a) $\chi_\alpha^{\mathfrak{p}'} = \chi_\alpha^{\mathfrak{p}}$
(b) $R_\alpha^{\mathfrak{p}'} = \langle R_\alpha, P, G_P \rangle$
(c) $\mathscr{E}_\alpha^{\mathfrak{p}'} = \{N[G_P] : N \in \mathscr{E}_\alpha \text{ and } P \in N \text{ and } N[G_P] \cap V = N\}$
(d) $D_\alpha(N[G_P]) = \big\{\{M[G_P] \in \mathscr{E}_\alpha : M \in Y \cap \bigcup\limits_{\beta<\alpha} \mathscr{E}_\beta^{\mathfrak{p}}\} : Y \in D_\alpha(N)\big\}$.

2.7 Claim. *Let $\mathfrak{p}, P$ be as in Definition 1.8.*
1) In Definition 2.6, $\mathfrak{p}^{V[G_P]}$ is a reasonable parameter in $V[G_P]$.
2) If $\mathfrak{p}$ is, in V, a ∂-winner (or non-∂-loser or ∂'-winner or non-∂'-loser), <u>then</u> $\mathfrak{p}^{V[G_P]}$ is so (in $V[G_P]$).
3) If $\mathfrak{p}$ is, in V, a $\partial_{\alpha,\beta}$-winner (or a non $\partial_{\alpha,\beta}$-lower or $\partial'_{\alpha,\beta}$-winner or $\partial'_{\alpha,\beta}$-non loser), <u>then</u> $\mathfrak{p}^{V[G_P]}$ is so in $V[G_p]$.

Proof. Straight (we use P is a proper forcing notion not adding reals).

* * *

2.8 Definition. Let $\mathfrak{p}$ be a reasonable parameter.
1) For $\alpha \le \beta < \ell g(\mathfrak{p})$, we say a forcing notion Q is $(\mathfrak{p}, \alpha, \beta)$-proper if:
$(\mathscr{P}(Q) \in \mathscr{H}(\chi_0^{\mathfrak{p}})$ and):

- $(*)$ for some $x \in \mathscr{H}(\chi_\beta)$ <u>if</u> $N \in \mathscr{E}_\beta, \{x, Q, \alpha\} \in N, p \in N \cap Q$ and $Y \in D_\alpha(N)$, <u>then</u> for some q we have:
  - (a) $p \le q \in Q$
  - (b) q is (N, Q)-generic
  - (c) version (1): for some $N' \in (\mathscr{E}_\alpha \cap N \cap Y) \cup \{N \cap \mathscr{H}(\chi_\alpha^{\mathfrak{p}})\}$ satisfying $\alpha = \beta \Rightarrow N' = N$ we have $q \Vdash_Q$ "$\mathscr{M}[\underset{\sim}{G}_Q, N', y^*] \cap Y \in D_\alpha(N')$" where $y^* = \langle x, p, Q \rangle$ hence necessarily q is (N', Q)-generic.

  (Version 2: similarly but $N' = N \restriction \mathscr{H}(\chi_\alpha^{\mathfrak{p}})$ and naturally we demand $\alpha \le_{\mathfrak{p}} \beta$).

2) We say Q is $(\mathfrak{p}, f)$-proper if $(\mathfrak{p}, f$ are as above and) for every $\alpha < \ell g(\mathfrak{p})$ and $\beta \in f(\alpha)$ we have Q is $(\mathfrak{p}, \alpha, \beta)$-proper.
3) We say Q is $\mathfrak{p}$-proper if Q is $(\mathfrak{p}, \alpha, \alpha)$-proper for $\alpha < \ell g(\mathfrak{p})$. We say Q is almost $\mathfrak{p}$-proper if Q is $(\mathfrak{p}, f)$-proper for some $f \in \mathscr{F}^{\mathfrak{p}}$ (as above).

2.9 Claim. *Assume $\mathfrak{p}$ is a simple reasonable parameter.*
1) If $\alpha' \le \alpha \le \beta \le \beta' < \ell g(\mathfrak{p})$ (for version 2 we demand $\alpha' \le_{\mathfrak{p}} \beta'$ and $\alpha \le_{\mathfrak{p}} \beta$) and Q is as $(\mathfrak{p}, \alpha, \beta)$-proper forcing notion, <u>then</u> Q is a $(\mathfrak{p}, \alpha', \beta')$-proper forcing notion.
2) Assume f, f' are in $\mathscr{F}^{\mathfrak{p}}$ and $f \le f'$. If Q is a $(\mathfrak{p}, f)$-proper forcing notion, <u>then</u> Q is a $(\mathfrak{p}, f')$-proper forcing notion.

Proof. Straight.

2.10 Discussion: We may like to consider $(\mathfrak{p}, f)$-proper for iterations which may add reals. Then we have to replace $D_\alpha(N)$ by a definition which is absolute enough, (and the non-loser versions have to be absolute enough).
It is natural to restrict ourselves to $\mathfrak{p}$-closed Y, see below.

2.11 Definition. Let $\mathfrak{p}$ be a simple reasonable parameter and $N \in \mathscr{E}_\alpha$. Now $Y \subseteq N$ is called $\mathfrak{p}$-closed if (see 3.9):

(a) $Y \subseteq N \cap \bigcup\limits_{\beta<\alpha} \mathscr{E}_\beta$

(b) if $M \in N \cap \mathscr{E}_\beta$ and $\beta < \alpha$ (hence $\beta \in \alpha \cap M \subseteq \alpha \cap N$) and $\gamma \in M \cap \beta$ and $M \cap \mathscr{H}(\chi^{\mathfrak{p}}_\gamma) \in \mathscr{E}_\gamma$ (this requirement is redundant if $\mathfrak{p}$ is simple or just $\gamma \leq_{\mathfrak{p}} \beta$),
then $M \cap \mathscr{H}(\chi^{\mathfrak{p}}_\gamma) \in Y \Leftrightarrow M \in Y$

(c) if $\beta < \alpha, M_\ell \in N \cap \mathscr{E}_\beta$ (hence $\beta \in \alpha \cap N$) and $M_\ell \subseteq M_{\ell+1}$ for $\ell < \omega$ and $M = \bigcup\limits_{\ell<\omega} M_\ell \in N \cap \mathscr{E}_\beta$, and even $\langle M_\ell : \ell < \omega\rangle \in N$, then $\bigwedge\limits_{\ell} M_\ell \in Y \Rightarrow M \in Y$.

We may consider

2.12 Definition. Let $\mathfrak{p}$ be a reasonable parameter and $\Xi \subseteq \{(\alpha,\beta) : \alpha \leq \beta < \ell g(\mathfrak{p})\}$ and we understand $N \in \mathscr{E}_\alpha \Rightarrow \Xi \cap ((\alpha+1) \times (\alpha+1) \in N)$ and let $\alpha_{\mathfrak{p}}(N) =$ the α such that $N \in \mathscr{E}^{\mathfrak{p}}_\alpha$ if it exists.

We say $\bar{Q}$ is an (κ_1, Ξ)-anti w.d.-iteration for $\mathfrak{p}$ as in clauses (a), (b), (d) of Definitions 1.8, 1.13 only we replace clause (d) by

$(d)_\Xi$ $[(\kappa,\Xi)$-anti w.d.$]$ like old clause (d) but for some $(\alpha,\beta) \in \Xi, N_0 \in N_1$ are countable elementary submodels of $(\mathscr{H}(\chi), \in), \mathfrak{p} \in N_0, N_0 \cap \mathscr{H}(\chi^{\mathfrak{p}}_\alpha) \in \mathscr{E}_\alpha, N_1 \cap \mathscr{H}(\chi^{\mathfrak{p}}_\beta) \in \mathscr{E}_\beta, \alpha \in N_0, \beta \in N_1$ and the rest as before.

We omit "for $\mathfrak{p}$" if $\mathfrak{p}$ is clear from the context.

2.13 Lemma. *Let $\mathfrak{p}$ be a reasonable parameter.*
1) If $\Xi \neq \emptyset$ and $\bar{Q}$ is an $NNR^0_{\aleph_0,\Xi}$-iteration, then $P_{\ell g(\bar{Q})} =$ $Lim(\bar{Q})$ does not add reals and is $(\mathfrak{p},\Xi)$-proper.

Proof. Easy.

As the sets $\mathscr{H}(\chi_\alpha)$ may change with forcing, we may prefer to use $\mathscr{E}_\alpha \subseteq [\chi_\alpha]^{\leq \aleph_0}$, for this we define:

2.14 Definition. 1) We call $\mathfrak{p}$ an o.b. (ordinal based) parameter if $\mathfrak{p} = (\bar\chi^{\mathfrak{p}}, \bar R^{\mathfrak{p}}, \bar{\mathscr{E}}^{\mathfrak{p}}, \bar D^{\mathfrak{p}})$ and for some ordinal α^* called $\ell g(\mathfrak{p})$ we have:

- (a) $\bar\chi = \langle \chi_\alpha : \alpha < \alpha^* \rangle, \chi_\alpha$ is a regular cardinal and $\mathscr{H}\big((\bigcup\limits_{\beta<\alpha} \chi_\beta)^+\big) \in \mathscr{H}(\chi_\alpha)$
- (b) $\bar R = \langle R_\alpha : \alpha < \alpha^* \rangle, R_\alpha$ an $n(R_\alpha)$-place relation on some bounded subset of χ_α (we could have asked "on χ_α", no real difference)
- (c) $\bar{\mathscr{E}} = \langle \mathscr{E}_\alpha : \alpha < \alpha^* \rangle, \mathscr{E}_\alpha \subseteq [\chi_\alpha]^{\le \aleph_0}$ is stationary
- (d) $\bar D = \langle D_\alpha : \alpha < \alpha^* \rangle$ and D_α is a function with domain $\mathscr{E}_\alpha$ and for each $a \in \mathscr{E}_\alpha, D_\alpha(a)$ is a family of subsets of a, closed under supersets, non-empty if $\alpha > 0$ (let $D^-_\alpha(a) = (D_\alpha(a))^- = \mathscr{P}(a) \backslash D_\alpha(a))$
- (e) let $\mathfrak{p}^{[\alpha]} = \langle \bar\chi \restriction \alpha, \bar R \restriction (\alpha+1), \bar{\mathscr{E}} \restriction \alpha, \bar D \restriction \alpha \rangle$
- (f) if $a \in \mathscr{E}_\alpha$ and $X \subseteq \mathscr{P}(a)$, <u>then</u>: $X \in D_\alpha(a) \Leftrightarrow X \cap \bigcup\limits_{\beta<\alpha} \mathscr{E}_\beta \in D_\alpha(a)$.

2) We say that an o.b. parameter $\mathfrak{p}$ is simple if

- (g) if $a \in \mathscr{E}_\alpha$ and $X \in D_\alpha(a)$ and $\beta \in \alpha \cap a$, <u>then</u> $a \cap \chi_\beta \in \mathscr{E}_\beta$.

3) For $\mathfrak{p}$ as above let $\mathfrak{q} =: \mathfrak{p}^V$ be defined by

$$\alpha^{*,\mathfrak{q}} = \alpha^{*,\mathfrak{p}}$$

and we define by induction on $\alpha < \alpha^{*,\mathfrak{p}}$:

$$\chi^{\mathfrak{q}}_\alpha = \chi^{\mathfrak{p}}_\alpha$$

$$R^{\mathfrak{q}}_\alpha = R^{\mathfrak{p}}_\alpha$$

$$\mathscr{E}^{\mathfrak{q}}_\alpha = \{N \prec (\mathscr{H}(\chi^{\mathfrak{q}}_\alpha), \in) : N \text{ is countable}, N \cap \chi^{\mathfrak{p}}_\alpha \in \mathscr{E}^{\mathfrak{p}}_\alpha, \mathfrak{q}^{[\alpha]} \in N\}$$

(note that $\mathfrak{q}^{[\alpha]}$ is well defined by the induction hypothesis)

$$\begin{aligned} D^{\mathfrak{q}}_\alpha(N) = \Big\{ Y' : & Y' \subseteq \bigcup_{\beta<\alpha} \mathscr{E}^{\mathfrak{q}}_\beta \text{ and for some } y \in N \cap \bigcup_{\beta<\alpha} \mathscr{H}(\chi^{\mathfrak{p}}_\beta) \\ & \text{and } Y \in D^{\mathfrak{p}}_\alpha(N \cap \chi^{\mathfrak{p}}_\alpha) \text{ we have} \\ & Y' \supseteq \{M : M \in N \cap \bigcup_{\beta<\alpha} \mathscr{E}^{\mathfrak{q}}_\beta \text{ and } y \in M \text{ and } M \cap \chi^{\mathfrak{p}}_\alpha \in Y\} \Big\}. \end{aligned}$$

4) For such $\mathfrak{p}$ we say $\bar{Q}$ is a NNR^0_κ-iteration for $\mathfrak{p}$ if it is an NNR^0_κ-iteration for $\mathfrak{p}^V$. We say $\mathfrak{p}$ is simple if $\mathfrak{p}^V$ is.
Similarly for $\eth$-winner, non-$\eth$-loser, etc.

2.15 Claim. *Assume $\mathfrak{p}$ is an o.b. [simple] parameter in the universe V.*
1) If $P \in \mathscr{H}(\chi^{\mathfrak{p}}_0)$ is a proper forcing notion (or at least preserve the stationarity of $\mathscr{E}^{\mathfrak{p}}_\alpha$ for each $\alpha < \ell g(\mathfrak{p})$), then $\Vdash_P$ "$\mathfrak{p}$ is an o.b. [simple] parameter".
2) If forcing with P adds no reals, then also $\eth$-winner, non-$\eth$-loser, etc., are preserved.
3) $\mathfrak{p}^V$ is a reasonable parameter.

2.16 Definition. Let $\mathfrak{p}$ be an o.b. parameter.
1) We say Q is an NNR^0_κ-forcing for $\mathfrak{p}$ or a $\mathfrak{p} - NNR^0_\kappa$-forcing notion when: ($Q$ is a forcing notion in a universe V and) if for some transitive class $V_0, \mathfrak{p} \in V_0$ and NNR^0_κ-iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$, we have $V_0^{P_\alpha} = V$, <u>then</u> we can let $\underset{\sim}{Q}_\alpha = Q$ and get $\bar{Q}'$, an NNR^0_κ-iteration (i.e. $V = V_0[G_\alpha], G_\alpha \subseteq P_\alpha$ is generic over V_0 and there is a P_α-name $\underset{\sim}{Q}_\alpha$ such that $\langle P_i, \underset{\sim}{Q}_i : i \leq \alpha \rangle$ is an NNR^0_κ- iteration and $Q = \underset{\sim}{Q}_\alpha[G_\alpha]$). In particular Q is proper and does not add reals.
2) If we omit "for $\mathfrak{p}$" we mean for any $\mathfrak{p}$ which makes sense. Alternatively, we can put a family of $\mathfrak{p}$'s.
3) We add "over x" if this holds whenever $x \in V_0$. We can use the same definition for other versions of NNR.
4) $Ax^\alpha_\lambda(\mathfrak{p}, \kappa, 0)$ means: if Q is an $\aleph_2$-e.c.c. (see [Sh:f, Ch.VII,§1]), ($\aleph_2$-pic if $\lambda = \aleph_2$; see [Sh:f, Ch.VIII,§2]), an NNR^0_κ-forcing notion for $\mathfrak{p}, \mathscr{I}_\beta$ a dense open subset of Q for $\beta < \beta^* < \lambda$ and $\underset{\sim}{S}_i$ is a Q_i-name of a stationary subset of ω_1 for $i < i^* < \alpha$, <u>then</u> for some directed $G \subseteq Q$ we have:
$\beta < \beta^* \Rightarrow G \cap \mathscr{I}_\beta \neq \emptyset, \underset{\sim}{S}_i[G] = \{\gamma < \omega_1 : (\exists r \in G)(r \Vdash_Q \text{"}\gamma \in \underset{\sim}{S}_i\text{"})\}$ is a stationary subset of ω_1.

Now we can deduce conclusions on preservation of being an NNR^0_κ-forcing notion for $\mathfrak{p}$ and consistency of axioms.

2.17 Conclusion. 1) if $\mathfrak{p}$ is o.b. parameter, non-$\eth'_{\rm id}$-loser, $\ell g(\mathfrak{p}) = \omega_1$ and $\bar{Q}$ is an CS iteration such that $\alpha < \ell g(\bar{Q}) \Rightarrow \Vdash_{P_\alpha}$ "$\underset{\sim}{Q}_\alpha$ is an NNR^0_κ-forcing notion for $\mathfrak{p}$", <u>then</u> $\bar{Q}$ is an NNR^0_κ-iteration for $\mathfrak{p}$ (note 1.9 applies).
2) Assume $CH + \mu = \mu^{<\mu} \geq \lambda$, if $\mathfrak{p}$ is a non$-\eth'_{\rm id}$-loser o.b. parameter, $\chi^{\mathfrak{p}}_0 > 2^\lambda$, <u>then</u> for some $\aleph_2$-e.c.c., (if $\lambda = \aleph_2, \aleph_2$-pic) NNR^0_κ-forcing notion $P, |P| = \mu$ we have

$\Vdash_P$ "$Ax_\lambda(\mathfrak{p}, \kappa, 0)$".
3) The parallel of 1.15 holds.

Proof. Straight.

§3 Examples: shooting a thin club

We would like here to see how some examples fit our framework. We already know that ($< \omega_1$)-proper forcings are $\mathfrak{p}$-proper (by 1.16). We first deal with a forcing notion for which the κ (in NNR^x_κ) is $< \aleph_0$ (in Definition 3.1). Second, we deal with shooting clubs of ω_1 running away from some $C_\delta \subseteq \delta = \sup(C_\delta)$ which are small (see [Sh:f, Ch.XVIII,§1]). Those are the most natural non-ω-proper forcing notions not adding reals, (though this may depend on the set theory).

The prototype for [Sh:b, Ch.VIII,§4], [Sh:f, Ch.VIII,§4] is 3.1(1) below.

3.1 Definition. 1) Let $\bar{C} = \langle (C_\delta, n_\delta) : \delta < \omega_1 \text{ limit}\rangle$ where C_δ is an unbounded subset of δ of order type ω and $1 \le n_\delta < \omega$ (e.g. $n_\delta = 1$). Let $\bar{u} = \langle u_\delta : \delta < \omega_1 \text{ limit}\rangle$, where $u_\delta \in [2n_\delta + 1]^{n_\delta}$, (if $n_\delta = 1$, then $u_\delta = \{k_\delta\}, k_\delta \in \{0,1,2\}$). Then we define

$$Q = Q_{\bar{C},\bar{u}} = \Big\{ f : \text{for some } \alpha < \omega_1, f \text{ is a function from } \alpha \text{ to } \omega \text{ such that for every limit ordinal } \delta \le \alpha, \text{ for some } k < 2n_\delta + 1, k \notin u_\delta \text{ and for every } i \in C_\delta \text{ large enough we have } f(i) = k \Big\}$$

ordered by inclusion.
2) Assume $\bar{C} = \langle C_\delta : \delta < \omega_1 \text{ limit}\rangle, C_\delta$ a closed subset of δ of order type $< \omega \times \delta$ and for $\delta_1 < \delta_2$ limit, $\sup(C_{\delta_1} \cap C_{\delta_2}) < \delta_1$, and for limit δ^* we have $\{C_\delta \cap \delta^* : \delta < \omega_1\}$ is countable. Assume further $\bar{\kappa} = \langle \kappa_\delta : \delta < \omega_1 \text{ limit}\rangle, \kappa_\delta \in \{2,3,\dots,\aleph_0\}, \bar{D} = \langle D_\delta : \delta < \omega_1\rangle, D_\delta$ is a family of subsets of $\mathrm{Dom}(D_\delta)$, such that the intersection of any $< \kappa_\delta$ of them is non-empty.

Let $\bar{f} = \langle f_{\delta,x} : \delta < \omega_1, x \in \mathrm{Dom}(D_\delta)\rangle$ satisfy $f_{\delta,x} : C_\delta \to \omega$ and $\bar{A} = \langle A_\delta : \delta < \omega_1\rangle$ satisfy $A_\delta \in D_\delta$. Then we define $Q = Q_{\bar{C},\bar{D},\bar{\kappa},\bar{f},\bar{A}}$ as

$$\Big\{ f : \text{for some } \alpha < \omega_1, f \text{ is a function from } \alpha \text{ to } \omega \text{ such that for every limit } \delta \le \omega \text{ for some } x \in A_\delta \text{ we have } f_{\delta,x} \subseteq^* f \text{ i.e. for every large enough } i \in C_\delta \text{ we have } f_{\delta,x}(i) = f(i) \Big\}$$

ordered by inclusion.

3.2 Claim. *1) The forcing notion Q from 3.1(1) is proper, does not add reals, and ia α-proper for $\alpha < \omega_1$ and even is $(< \omega_1)$-proper, also $\mathbb{D}$-complete for some simple 2-completeness system, hence causes no problem to the demand (d) in Definition 1.9 which means:*

- (∗) *if $\bar{Q}$ is a CS iteration, $\ell g(\bar{Q}) = \alpha + 1, \bar{Q} \restriction \alpha$ is NNR^0_2-iteration and $\Vdash_{P_\alpha}$ "$\underset{\sim}{Q}_\alpha = Q_{\bar{c},u}$", $(\bar{c}, \bar{u})$ as above $\in V$ <u>then</u> $\bar{Q}$ satisfies demand ((a), (b) and) (d)) of Definition 1.9.*

2) Similarly for the forcing notion Q from 3.1(2).

Proof. 1) See [Sh:b, Ch.VIII,§4] or [Sh:f, Ch.VIII,§4].
2) Similarly.

3.3 Definition. Assume $\bar{C} = \langle C_\delta : C_\delta < \omega_1 \rangle, C_\delta$ an unbounded subset of the limit ordinal δ (think of the case C_δ of order type $< \delta$ but not necessarily). Let

$$Q_{\bar{C}} = \Big\{ c : \text{for some } \alpha < \omega_1, c \text{ is a closed subset of } \alpha \text{ and for every limit ordinal } \delta \leq \alpha \text{ we have : } \delta = \sup(c \cap \delta) \Rightarrow c \cap C_\delta \text{ bounded in } \delta \Big\}$$

$$c_1 < c_2 \text{ iff } c_1 \text{ is an initial segment of } c_2.$$

Remark. 1) See [Sh:f, Ch.XVIII,§1], note $Q_{\bar{C}}$ may be not ω-proper. We first deal with the simple case: $\text{otp}(C_\delta) = \omega$.
2) Note that $c \in Q_{\bar{c}}$ is a closed subset of α, but not necessarily a closed subset of ω_1.

3.4 Claim. *Assume $\mathfrak{p}$ is a simple reasonable parameter, $\bar{C}$ as in 3.3, $\bigwedge_\delta \text{otp}(C_\delta) = \omega$, let $f \in \mathscr{F}^{\mathfrak{p}}$ be $f(0) = 0, f(\beta) = 1 + \beta$, <u>then</u> $Q_{\bar{C}}$ is $(\mathfrak{p}, f)$-proper.*

We shall prove more in 3.6 below.

3.5 Remark. 1) Note that if $\bigwedge_\delta \text{otp}(C_\delta) < \delta$, we can split the analysis by, for each ordinal $\gamma < \omega_1$ such that $S_\gamma = \{\delta : \text{otp}(C_\delta) = \gamma\}$ is stationary, that is restricting ourselves to $\{N : N \cap \omega_1 \in S_\gamma\}$.

3.6 Claim. *1) Assume*

- (a) $\mathfrak{p}$ *is a simple reasonable parameter*
- (b) $\mathfrak{p}$ *is non-*$\eth_f$*-loser,* $f \in \mathscr{F}^{\mathfrak{p}}$
- (c) $\gamma(*) < \omega_1$
- (d) $\bar{C} = \langle C_\delta : \delta < \omega_1 \rangle, C_\delta \subseteq \delta = \sup(C_\delta)$
- (e) $\text{otp}(C_\delta) \le \omega^{\gamma(*)}$ *for every* δ

Define $g \in \mathscr{F}^{\mathfrak{p}}$ *by* $g(0) = 0, g(1) = f(1) + \gamma(*), g(\alpha+1) = f(g(\alpha)) + \gamma(*) + 1$ *(and necessarily, for limit* $\alpha, g(\alpha) = \sup\limits_{\beta<\alpha} g(\beta)$*).*
Then for every $\alpha < \ell g(\mathfrak{p})$*, the forcing notion* $Q = Q_{\bar{C}}$ *is* $(\mathfrak{p}, \alpha, g(\alpha))$*-proper (version 1).*
2) We can in part (1) get "version 2" when: if $g(\delta) = \delta, N \in \mathscr{E}_\alpha, C \subseteq \omega_1 \cap N = \sup(C)$, $\text{otp}(C) \le \omega^{\gamma(*)}$ *(for example* $C = C_\alpha \cap N$ *for some* α*) and* $Y \in D_\alpha(N)$ *then* $Y' = \{M \in Y : M \cap \omega_1 \notin C\} \in D_\alpha(N)$*, which is a mild condition.*
3) If we weaken clause (b) *to* $(b)_{f,g}$ *below, then for* $\alpha < \ell g(\mathfrak{p})$*, the forcing notion* $Q_{\bar{C}}$ *is* $(\mathfrak{p}, \alpha, f(\gamma + \alpha))$*-proper, where*
$(b)_{f,g}$ $f \in \mathscr{F}^{\mathfrak{p}}$ *and* $f(f(\alpha)) = f(\alpha)$ *for every* $\alpha < \ell g(\mathfrak{p})$ *and* $\mathfrak{p}$ *is a non-*$\eth'_f$*-loser.*

Proof. First observe

- $(*)$ for each $\alpha < \omega_1$ the set $\mathscr{I}^*_\alpha = \{p \in Q_{\bar{C}} :$ there is $\beta \in p$ which is $\ge \alpha\}$ is a dense subset of Q.

1) Similar to the proof of 2) noting version (2) is harder on the chooser, and using the extra freedom we can avoid the need for the extra assumption from (2).
2) We prove this by induction on α, let $\beta = g(\alpha)$.
Let $N \in \mathscr{E}_\beta$ be countable, $p \in N \cap Q_{\bar{C}}$ (and $Q_{\bar{C}}, \alpha, \beta \in N$) and $Y \in D_\beta(N)$ be given. Let $\delta = \delta_N =: N \cap \omega_1$.

Case 1: $\alpha = 0$.
Comprehending the demand in Definition 2.8(1) it just means "$Q_{\bar{C}}$ is proper". Let $\langle \mathscr{I}_n : n < \omega \rangle$ list the dense open subsets of $Q_{\bar{C}}$ that belongs to N and we shall define by induction on n, a condition p_n such that

- (i) $p_0 = p, p_n \in N$
- (ii) $p_n \le p_{n+1} \in \mathscr{I}_n$
- (iii) the set $p_{n+1} \cup \{\sup p_{n+1}\} \backslash (p_n \cup \{\sup p_n\})$ is disjoint to C_δ.

If we succeed to carry out the induction clearly we are done as $\bigcup\limits_{n<\omega} p_n$ is as required, noting for each $\alpha < N \cap \omega_1, \mathscr{I}^*_\alpha \in \{\mathscr{I}_n : n < \omega\}$, hence $(\exists \beta \in (\bigcup\limits_{n<\omega} p_n))[\alpha \le \beta < \omega_1]$. Also there is no problem to choose p_0. So assume p_n has been chosen and we shall choose p_{n+1} as required. There is a function $F_n \in N$ with domain $Q_{\bar{C}}$ such that: $q \in Q_{\bar{C}} \Rightarrow q \le F_n(q) \in \mathscr{I}_n$.

For $\alpha < \omega_1$ let $q^{[\alpha]} = q \cup \{\sup(q)\} \cup \{\sup(q) + 1 + \alpha\}$, so clearly $q \le q^{[\alpha]} \in Q_{\bar{C}}$ and the function $(q, \alpha) \mapsto q^{[\alpha]}$ belongs to N. Define a function $H : \omega_1 \to \omega_1$ by $H(\alpha) = \sup(F_n(p_n^{[\alpha]}))$, clearly it is well defined, belongs to N and let $C =: \{\beta < \omega_1 :$ β a limit ordinal, moreover $\omega\beta = \beta$ and $(\forall \alpha < \beta)(H(\alpha) < \beta)$ and $\sup(p_n) < \beta\}$, so C is a club of ω_1 which belongs to N and $\gamma(*) \in N$ hence we can find $\beta^* \in C$ such that $\mathrm{otp}(\beta^* \cap C)$ is divisible by $\omega^{\gamma(*)}$, but $\mathrm{otp}(C_\delta \cap \beta^*) < \omega^{\gamma(*)}$ hence for some $\beta \in C$ we have $\sup(C_\delta \cap \beta) < \beta$. Let $p_{n+1} = F_n(p^{[\sup(C_\delta \cap \beta)+1]})$; it is as required.

<u>Case 2</u>: $\alpha = 1$.

Easily $Y' =: \{M \in Y : M \cap \omega_1 \notin C_\delta\} \in D_{g(1)}(N)$ as $\beta \ge g(1) = f(1) + \gamma(*)$, just prove this by induction on $\gamma(*)$. Let $\langle \mathscr{I}_n : n < \omega \rangle$ list the dense open subsets of $Q_{\bar{C}}$ which belong to N and let $\delta = \bigcup\limits_{n<\omega} \alpha_n$ and $\alpha_n < \alpha_{n+1} < \delta$. We now simulate a strategy for the challenger in the game $\partial_{\alpha,\beta}(N)$, together with choosing in the n-th move, (in the end of it) also the challenger chooses $p_{n+1} \in Q_{\bar{C}} \cap N$ such that $p_0 = p, p_n \le p_{n+1} \in \mathscr{I}_n$ and during the play letting $Z_n = \emptyset$ (in fact also the chooser has to use $Y_n = \emptyset$) and p_{n+1} is $(M_n, Q_{\bar{C}})$-generic and $\sup(p_{n+1}) > \alpha_n$ and the set $(p_{n+1} \cup \{\sup p_{n+1}\}) \backslash (p_n \cup \{\sup p_n\})$ is disjoint to C_δ. This is possible by Case 1 and its proof because $M_n \cap \omega_1 \notin C_\delta$ which holds as $M_n \in Y'$. As this is a legal strategy for the challenger, so it cannot be a winning strategy hence for some such play the chooser wins hence $\{M_n : n < \omega\} \in D_1(N)$, remember $\alpha = 1$. Now $q = \bigcup\limits_{n<\omega} p_n$ is well defined, and $\sup(q) = \delta$ and $q \cap C_\delta \subseteq p \cup \{\sup(p)\}$ and $q \Vdash_{Q_{\bar{C}}}$ "$\{M_n : n < \omega\} \subseteq \mathscr{M}[\underset{\sim}{G}_{Q_{\bar{C}}}, N]$", so q is as required as the chooser has won the play.

<u>Case 3</u>: $\alpha > 1, \alpha$ successor.

Similar to Case 2 only we use the induction hypothesis instead of using Case 1.

<u>Case 4</u>: α a limit ordinal.
Similar to Case 2.
Easy. $\square_{3.6}$

3.7 Definition. 1) Assume

- (a) $S \subseteq \omega_1$ is stationary
- (b) $f \in {}^{\omega_1}(\omega_1)$
- (c) $\bar C = \langle C_\delta : \delta < \omega_1 \rangle$
- (d) C_δ is an unbounded subset of δ.

We define when "$\bar C$ obeys f on S" for $f \in {}^{\omega_1}\omega_1$ a $(\mathscr{D}_{\omega_1} + S, \gamma)$-th function (see part (2)) by induction on γ. For f being $(\mathscr{D}_{\omega_1} + S, \gamma)$-function, $\gamma < \omega_1$ this means $\{\delta \in S : \text{otp}(C_\delta) \le \omega^{1+f(\delta)}\} = S \bmod \mathscr{D}_{\omega_1}$. In general it means that for some $g : \omega_1 \to \omega_1$ and pressing down function on h on S, for every $\zeta < \omega_1$ for which $h^{-1}\{\zeta\}$ is stationary, for some $\beta < \gamma$ and f_β, a $(\mathscr{D} + h^{-1}\{\zeta\}, \beta)$-th function, we have $\langle C_{g(\delta)} \cap \delta : \delta \in h^{-1}\{\zeta\} \rangle$ obeys f_β.
2) We say $f \in {}^{\omega_1}(\omega_1)$ is a $(\mathscr{D}_{\omega_1} + S, \gamma)$-th function when: $S \Vdash_{(\mathscr{D}^+_{\omega_1}, \supseteq)}$ "in $V[\underset{\sim}{G}]$, $\{x \in V^{\omega_1}/\underset{\sim}{G} : V^{\omega_1}/G \models x \text{ an ordinal} < f/\underset{\sim}{G}\}$ has order type γ".

3.8 Claim. *Assume*

- (a) $\mathfrak{p}$ *is a simple reasonable parameter,* $\ell g(\mathfrak{p})$ *of uncountable cofinality*
- (b) $S \in \mathscr{D}^+_{\omega_1}$ *and* $N \in \bigcup_\alpha \mathscr{E}^{\mathfrak{p}}_\alpha \Rightarrow N \cap \omega_1 \in S$
- (c) $\mathfrak{p}$ *is a non-*$\eth_{\alpha,\alpha}$*-loser (or just non* $\eth'_{\alpha,\alpha}$*-loser) for all* $\alpha \in C^*, C^*$ *a club of* $\ell g(\mathfrak{p})$,
- (d) $\bar C$ *obeys* f *on* S *which is a* $(\mathscr{D}_{\omega_1} + S, \gamma)$*-function*
- (e) $0 < \text{ Min}(C), g(\alpha) = \text{ Min}(C^* \backslash \alpha)$.

<u>Then</u> $Q_{\bar C}$ *is* $(\mathfrak{p}, g)$*-proper.*

Proof. Similar.

* * *

Another example (which could have been done in [Sh:f, Ch.XVIII]) is:

3.9 Definition. 1) We say $\bar{\mathscr{D}} = \langle \mathscr{D}_\delta : \delta < \omega_1 \text{ limit} \rangle$ is an ω_1-filter-sequence if:

- (a) $\mathscr{D}_\delta$ is a filter on δ, containing the co-bounded subsets
- (b) $\mathscr{D}_\delta$ is a P-filter and some $C_\delta \in D_\delta$ has order type ω (P-filter means that $\mathscr{D}_\delta$ contains all co-finite sets and if $A_n \in D_\delta$ for $n < \omega$ then for some $A \in D_\delta$ we have $n < \omega \Rightarrow |A \backslash A_n| < \aleph_0$) (can generalize as in 3.1(2) but did not)

(c) for every club $C \subseteq \omega_1$ and $\alpha < \omega_1$, the set $A^\alpha_C[\bar{\mathscr{D}}]$ is stationary, where, by induction on α we define:
$A^\alpha_C[\bar{\mathscr{D}}] = \{\delta < \omega_1 : \delta$ is a limit ordinal, $\delta \in C$ and for every $\beta < \alpha$ we have $\delta = \sup(\delta \cap A^\beta_C[\bar{\mathscr{D}}])$, moreover $\delta \cap A^\beta_C[\bar{\mathscr{D}}] \in \mathscr{D}_\delta\}$.

2) We say a reasonable parameter $\mathfrak{p}$ obeys $\bar{\mathscr{D}}$ <u>if</u> for each $\alpha < \ell g(\mathfrak{p})$ and $N \in \mathscr{E}_\alpha$ we have: $\bar{\mathscr{D}} \in N$ and we have

$$\begin{aligned}\mathscr{D}^{\mathfrak{p}}_\alpha(N) = \Big\{Y : & Y \subseteq N \cap \bigcup_{\beta<\alpha} \mathscr{E}_\beta \text{ is closed (see 2.11) and if} \\ & \alpha > 0 \text{ then there are} \\ & \bar{\beta} = \langle \beta_n : n < \omega\rangle, \bar{M} = \langle M_n : n < \omega\rangle \text{ satisfying} \\ & (a) \quad \beta_n \in N \cap \alpha \text{ and} \\ & (b) \quad \text{either } \bigwedge_n [\alpha = \beta_n + 1] \\ & \qquad \underline{\text{or}}\ \beta_n < \beta_{n+1}, \sup_{n<\omega} \beta_n = \sup(\alpha \cap N) \\ & (c) \quad M_n \in Y \cap \mathscr{E}_{\beta_n}, M_n \in M_{n+1}, \\ & \qquad \bigcup_{n<\omega} M_n = N \cap \bigcup_{\beta \in \alpha \cap N} \mathscr{H}(\chi_\beta) \\ & (d) \quad \{M_n \cap \omega_1 : n < \omega\} \in \mathscr{D}_{N \cap \omega_1}\Big\}.\end{aligned}$$

3) We say a forcing notion Q is a $\bar{\mathscr{D}} - NNR^0_\kappa$-forcing if for every reasonable parameter $\mathfrak{p}$ which obeys $\bar{\mathscr{D}}$ we have: Q is an NNR^0_κ-forcing over $\mathfrak{p}$ (see 2.16).
4) For a P-filter $\mathscr{D}$, we say a reasonable parameter $\mathfrak{p}$ obeys $\mathscr{D}$ if:
for every $N \in \mathscr{E}_\alpha$

$$\begin{aligned}D^{\mathfrak{p}}_\alpha(N) = \Big\{Y : & Y \subseteq N \cap \bigcup_{\beta<\alpha} \mathscr{E}_\beta \text{ is closed (see 2.11) and if } \alpha > 0 \text{ then} \\ & \text{for some } \bar{\beta}, \bar{M} \text{ satisfying clauses } (a), (b), (c) \\ & \text{of part (2) and} \\ & (d)' \qquad \{n : M_n \in Y\} \in \mathscr{D} \\ & \qquad\qquad (*)\mathscr{D} \text{ is a } P\text{-filter on } \omega\Big\}.\end{aligned}$$

5) In the parts 1) - 4) above we may replace the word filter by ultrafilter if the $\mathscr{D}$'s are ultrafilter (so we have "ω_1-ultrafilter sequence").

3.10 Claim. *1) If $\diamondsuit_{\aleph_1}$ (or much less), then there is an ω_1-ultrafilter sequence.*
2) If $\bar{\mathscr{D}}$ is an ω_1-filter sequence and $\langle (\chi_\alpha, \mathscr{E}_\alpha) : \alpha < \alpha^* \rangle$ is a in Definition 1.1, then there is a reasonable parameter $\mathfrak{p}$ of length ω_1 obeying $\bar{\mathscr{D}}$ which is a non-$\Game$-loser $\chi^{\mathfrak{p}}_\alpha = \chi_\alpha, \mathscr{E}^{\mathfrak{p}}_\alpha = \mathscr{E}_\alpha$.
3) If $\diamondsuit_{\aleph_1}$ and $\langle (\chi_\alpha, \mathscr{E}_\alpha) : \alpha < \alpha^* \rangle$ is as in Definition 1.1 and $\mathscr{D}$ is a P-filter on ω, then some reasonable parameter $\mathfrak{p}$ is P-filter like, non-$\Game_{\mathrm{id}}$-loser with $\chi^{\mathfrak{p}}_\alpha = \chi_\alpha, \mathscr{E}^{\mathfrak{p}}_\alpha = \mathscr{E}_\alpha, \alpha^{*,\mathfrak{p}} = \alpha^*$. Similarly for ultrafilters.
4) Instead of $\diamondsuit_{\aleph_1}$ it is enough to assume CH and that for some $\langle C_\delta : \delta < \omega_1 \text{ limit}\rangle$ and normal filter D on ω_1 and for every club C of $\omega_1, \{\delta : \delta > \sup(C_\delta \backslash C)\} \in D$.

Proof. Straight (in part (4), we define the C_δ's simultaneously).

3.11 Claim. *1) If $\mathfrak{p}$ is a reasonable parameter obeying $\mathscr{D}$, a P-filter on ω [or P-ultrafilter on ω], then for some $\bar{\mathscr{D}}, \bar{\mathscr{D}}$ is an ω_1-filter-sequence [or ω_1-ultrafilter-sequence] and $\mathfrak{p}$ obeys $\bar{\mathscr{D}}$.*
2) If $\mathfrak{p}$ is a P-point filter (or ultrafilter), then $\mathfrak{p}$ is a non-$\Game$-loser.

Proof. Should be clear.

3.12 Remark. We may consider higher order types than ω.

3.13 Claim. *1) Assume*

- (a) $\bar{C} = \langle C_{\delta,\ell} : \ell < k_\delta, \delta < \omega_1 \text{ limit}\rangle, 1 + \kappa \leq k_\delta \leq \omega, C_{\delta,\ell}$ *is a closed unbounded subset of δ and $\ell < m < k_\delta \Rightarrow C_{\delta,\ell} \cap C_{\delta,m} = \emptyset$*
- (b) $Q = Q_{\bar{C}} = \{C : C$ *is a closed bounded subset of ω_1 such that for every limit $\delta < \sup(C)$ and for every $\ell < k_\delta$ except $< 1 + \kappa$ many, $\delta > \sup(C \cap C_{\delta,\ell})\}$*
- (c) *$\mathfrak{p}$ obeys the P-ultrafilter $\mathscr{D}$ and is a reasonable parameter.*

Then Q is a $\mathfrak{p} - NNR^0_\kappa$ forcing notion.
2) In part (1) if we add: $N \in \mathscr{E}_\alpha$ & $D^{\mathfrak{p}}_\alpha(N) = \{Y : \{n : M_n \in Y\} \in \mathscr{D}_N\}, \mathscr{D}_N$ a P-point ultrafilter (see above) and $\ell < k_\delta \Rightarrow \{n < \omega : M_n \cap \omega_1 \in C_{\delta,\ell}\} = \emptyset$ mod $\mathscr{D}_N$, then we can allow $C_{\delta,0} = C_{\delta,1}$.
3) Assume

(a) D_δ *is a family of subsets of* $Dom(D_\delta)$*, the intersection* Y *of any* $< 1+\kappa$ *of them satisfies*

(∗) $\exists n(\exists y_1,\ldots,y_n \in Y)[\delta > \sup(\bigcap_{\ell=1}^{n} C_{\delta,y_\ell})]$

(b) $\bar C = \langle C_{\delta,x} : x \in$ *Dom*(D_δ) *and* δ *is a limit ordinal* $< \omega_1\rangle$

(c) $\langle C_{\delta,x} : x \in$ *Dom*$(D_\delta)\rangle$ *is a sequence of pairwise disjoint subsets of* δ

(d) $\bar X \in \prod_{\delta<\omega_1} Dom(D_\delta)$

(e) $Q_{\bar C,\bar X,\bar D} = \{C : C$ *is a closed bounded subset of* ω_1 *such that for every limit* $\delta \le \sup(C)$ *we have* $(\exists x \in X_\delta)(\delta > \sup(C \cap C_{\delta,x}))\}$
ordered by: end extension

(f) $\bar{\mathscr D}$ *is a* P*-ultrafilter sequence.*

<u>*Then*</u> *Q is a* $\mathfrak p - NNR^0_\kappa$ *forcing notion* $\mathfrak p$ *which obeys* $\bar{\mathscr D}$*.*

Proof. Straight.
1) So let $V = V_0^{P_\alpha}, \Vdash_{P_\alpha}$ "$Q_\alpha = Q_{\underset{\sim}{\bar C}}$ is as above" and in $V_0, N_0 \in \mathscr E_0, N_0 \in N_1 \in \mathscr E_\alpha = \{M_n : n < \omega\}, D_{N_0}$ as in Definition 3.13(2), $p \in P_{\alpha+1} \cap N, G_m \subseteq N_1 \cap P_\alpha$ is generic over N_1 for $m < k < 1+\kappa$ (so k is now fixed). So for each ℓ we have $\underset{\sim}{C}_{N_0\cap\omega_1,\ell}[G_m]$ is a closed subset of δ and for $\ell_1 < \ell_2 < k_{\delta_{N_0}}$ we have $\underset{\sim}{C}_{N_0\cap\omega_1,\ell_1}[G_m] \cap \underset{\sim}{C}_{N_0\cap\omega_1,\ell_2}[G_m] = \emptyset$ so for some $\ell(m) \in \{0,1,\ldots,k_{\delta_{N_0}}-1\}$ we have $\ell \ne \ell(m) \Rightarrow \underset{\sim}{C}_{N_0\cap\omega_1,\ell}[G_m] = \emptyset \mod D_{N_0}$. Now let $B = \{n :$if $\ell < k_\delta, \ell \notin \{\ell(m) : m < k\}$ and $\ell < n$ then $M_n \cap \omega_1 \notin \underset{\sim}{C}_{N_0\cap\omega_1,\ell}[G_m]$ for $m < k$ and $p \in M_n\}$ belongs to D_{N_0}. Clearly, without loss of generality $\langle M_n : n < \omega\rangle \in N_1, D_{N_0} \in N_1, B \in N_1$. Let $B = \{n_i : i < \omega\}, n_i$ increasing with i and $\langle \mathscr I_n : n < \omega\rangle$ list the dense open subsets of $P_{\alpha+1}$ which belongs to N_0. We can choose p_i by induction on $i < \omega$ such that:

$$p_i \in N_{n_i} \cap P_{\alpha+1}$$

$$p_i \restriction \alpha \in \bigcap_{m<k} G_m$$

$$p_i \in \bigcap\{\mathscr I_n : n < n_i, \mathscr I_n \in N_{n_i} \text{ and } i > 0\}$$

$$p \le p_i$$

$$p_i \le p_{i+1}$$

$$p_{i+1} \backslash p_i \text{ is disjoint to } \cup \{\underset{\sim}{C}_{N_0 \cap \omega_1, \ell}[G_m] : \ell < k_0, \ell < n_i \text{ and } m < k \Rightarrow \ell \ne \ell(m)\}.$$

This is possible as

$$\otimes \left(\bigcup_{\substack{m<k \\ \ell<k \ \& \ \ell \ne \ell(m)}} \underset{\sim}{C}_{N_0 \cap \omega_1, \ell}[G_m] \right) \cap \omega_1 \cap M_n \text{ is a bounded subset of } M_n \cap \omega_1.$$

2), 3), 4) Similarly. $\square_{3.13}$

3.14 Remark. 1) The forcing notion from 3.13(1) is $NNR^0_{\aleph_0}$-forcing notion for every $\mathfrak{p}$, non-$\eth_{\mathrm{id}}$-loser.
2) If $\mathfrak{p}$ is a reasonable parameter obeying $\bar{\mathscr{D}}$, an ω_1-filter sequence, then any $(< \omega_1)$-proper forcing is $\mathfrak{p}$-proper.

Proof. Similar to earlier proofs.

§4 Second preservation of not adding reals

We shall concentrate on the simple case.

4.1 Definition. Let $\mathfrak{p}$ be a reasonable parameter. We say that $\bar{Q}$ is a $\mathfrak{p} - NNR^1_\kappa$ iteration (where $2 \le \kappa \le \aleph_0$ as we omit $\kappa = \aleph_1$ for convenience) if:

- (a) $\mathrm{cf}(\ell g(\mathfrak{p})) > \ell g(\bar{Q})$ (for simplicity)
- (b) $\bar{Q}$ is a countable support iteration of proper forcing notions
  [in [Sh 311] we will relax the requirement that each iterand is proper]
  such that $i < \ell g(\bar{Q}) \Rightarrow P_i$ adds no reals
  (follows by other parts (close (d)) even for $P_{i+1}, i < \ell g(\bar{Q})$)
- (c) [long properness] for each i, for some $f_i \in \mathscr{F}^{\mathfrak{p}}_{\mathrm{club}}$ we have $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is $(\mathfrak{p}, f_i)$-proper, f_i increasing continuous, see Definition 2.8(2) + 2.8(4)
- (d) [κ-anti w.d.] if $f_i(\alpha) \le \beta < \ell g(\mathfrak{p}), \{\bar{Q}, i, \alpha, \beta\} \in N_0 \in \mathscr{E}_\alpha, N_0 \in N_1 \in \mathscr{E}_\beta, k < 1+\kappa, j = i+1$ and for $\ell = 0, 1, \dots, k-1$ we have $q_\ell \in P_i$ is (N_1, P_i)-generic and is (N_0, P_i)-generic, $q_\ell \Vdash$ "$\underset{\sim}{G}_{P_i} \cap N_1 = G^\ell$", $G^\ell \cap N_0 = G^*, p^* \in P_j \cap N_0$ satisfies $p^* \restriction i \in G^{**}$ and lastly $Y =: \bigcap\limits_{\ell<k} \mathscr{M}[G^\ell, N_1, y^*] \in D_\beta(N_1)$, <u>then</u> for some $G^{**} \in N_1, G^{**} \subseteq N_0 \cap P_j$ is generic over $N_0, G^* \subseteq G^{**}$ and $p^* \in G^{**}$ and $\bigwedge\limits_{\ell<k} \bigvee\limits_{r \in G^\ell} r \Vdash_{P_i}$ "$G^{**} \restriction \underset{\sim}{Q}_i (= \{p(i) : p \in G^{**}\})$ has an upper bound in $\underset{\sim}{Q}_i$".

4.2 Remark. 1) κ is the amount of "$\mathbb{D}$-completeness", in other words what versions of weak diamond we kill by our iteration.

2) <u>Question</u>: Why do we ask for $f_i \in V$ and not a P_i-name $\underset{\sim}{f}_i$ of such a function?

<u>Answer</u>: If $i < \ell g(\bar{Q}) \Rightarrow P_i \models$ $\mathrm{cf}(\ell g(\mathfrak{p}))$-c.c., then we can find $f'_i \in \mathscr{F}^{\mathfrak{p}}, f_i \ge \underset{\sim}{f}_i$; so as we are assuming, for conveniency $\mathrm{cf}(\ell g(\mathfrak{p})) > |P_\alpha|$ there is no point at present for f_i to be a P_i-name.
2A) In clause (d) we have implicitly used:

- $(*)$ if $\alpha \le \beta' < \beta$ then clause (d) for (α, β) and k implies clause (d) for (α, β') and k.

This holds by clause (i) of Definition 1.1.
We could replace f_i by a club E_i of $\ell g(\mathfrak{p})$, letting $f_i(\alpha) = E_i \backslash \alpha$.

3) No real reason to "use some κ", but also no real reason to use 2. For each ℓ to find G^{**} means essentially "forcing with Q_i adds no reals". The point is the common solution.
4) In clause (c) for a club C of $\ell g(\mathfrak{p})$ we catch our tail, that is $f_i(\alpha) \cap C = C \backslash \alpha$ for a club of $\alpha < \ell g(\mathfrak{p})$. We could use less, no real point now.
5) In clause (d) much of the freedom/variation will be due to the decision how "similar" are $\langle G^\ell : \ell < k \rangle$ such that G^{**} exists. Here we demand

(α) $Y \in D_\beta(N_1)$.

In [Sh:f, Ch.VIII,§2] it is essentially required that

(β) $G^0 \times G^1 \times \cdots \times G^{k-1} \subseteq (P_i \times \cdots \times P_i) \cap N_1$ (k times) is generic over N_1.

In [Sh:f, Ch.V]

(γ) the common Y is a predetermined increasing sequence of models.

We have a trade-off; clause (β) makes demand (d) in 4.1 easier, but the parallel of (c) harder compared to clause (α).
This explains why the present work doesn't supercede Ch.XVIII,§2.
6) It is harder to win with a "slower" function, so the assumption above is the strongest possible - although practically makes no difference, probably.

4.3 Claim. *Let $\bar{Q}$ be a $\mathfrak{p} - NNR^1_\kappa$ iteration (see Definition 4.1), $\kappa \in \{2, \aleph_0\}$ and $\mathfrak{p}$ is a reasonable parameter and $\mathfrak{p}$ is a $\eth_f$-winner for some $f \in \mathscr{F}^{\mathfrak{p}}_{\rm club}$ or at least $\eth'_f$-non loser.*
1) Forcing with $P_{\ell g(\bar{Q})} = Lim(\bar{Q})$ does not add reals (so consequently no ω-sequences, as we are assuming properness).
2) If $i \le j \le \ell g(\bar{Q})$, <u>then</u>

$(b)'$ P_j/P_i is proper

$(c)'$ P_j/P_i is $(\mathfrak{p}, f_{i,j})$-proper, where $f_{i,j} \in \mathscr{F}^{\mathfrak{p}}_{\rm club}$ is increasing continuous and is computable from the $f_\varepsilon \in \mathscr{F}^{\mathfrak{p}}$ for $\varepsilon \in [i, j)$,

$(d)'$ we have the parallel of clause (d), this time without assuming that $j = i+1$.

Proof. By induction on $\ell g(\bar{Q})$. For notational simplicity we assume

$\boxtimes$ all f_i are even in $\mathscr{F}^{\mathfrak{p}}_{\rm end}$ so consider them as functions from $\ell g(\mathfrak{p})$ to $\ell g(\mathfrak{p})$ (increasing continuous), and we demand also the $f_{i,j}$ are like that, increasing continuous and moreover $f_{i,j}(f_{i,j}(\alpha)) = f_{i,j}(\alpha)$ and they are $\ge f^*, f^* \in \mathscr{F}^{\mathfrak{p}}_{\rm end}$ increasing continuous, $\mathfrak{p}$ is $\eth_{f^*}$-winner or at least $\eth'_{f^*}$-non-loser.

Case 1: $\ell g(\bar{Q}) = 0$

Trivial.

Case 2: $\ell g(\bar{Q}) = i(*) + 1$

Part (1): $P_{i(*)}$ adds no reals by the induction hypothesis, $\underset{\sim}{Q}_{i(*)}$ adds no reals by clause (d) in Definition 4.1, hence $P_{i(*)+1} = P_{i(*)} * \underset{\sim}{Q}_{i(*)}$ adds no reals.

Part (2): Clause $(b)'$ is known, (namely P_j/P_i is proper).

Clause $(c)'$:

Given $i \le j \le \ell g(\bar{Q})$ if $j < i(*) + 1$ the conclusion follows by the induction hypothesis. So assume $j = i(*) + 1$. If $i = j$, the required demand is trivial, so assume $i < j$. If $i = i(*)$, use clause (c) of Definition 4.1 for i. So assume $i < i(*)$. Let $f_{i,j,0} = f_{i(*)}, f_{i,j,m+1} = f_{i(*)} \circ f_{i,j(*)} \circ f_{i,j,m}$ and $f_{i,j}(\alpha) = \sup\limits_{m<\omega} f_{i,j,m}(\alpha)$. Check that they are as required in $\boxtimes$. For proving "P_j/P_i is $(\mathfrak{p}, f_{i,j})$-proper" assume

- (a) $N \prec (\mathscr{H}(\chi), \in)$ is countable
- (b) $\{\bar{Q}, i, j, \alpha, \beta, f_{i,i(*)}, f_{i(*)}, f_{i,j}\} \in N$
- (c) $\alpha \le f_{i,j}(\alpha) \le \beta < \ell g(\mathfrak{p})$
- (d) $q \in P_i$ is (N, P_i)-generic
- (e) $p \in N \cap P_j, p \restriction i \le q$
- (f) $Y \in D_\beta(N)$
- (g) $q \Vdash$ "$Y \subseteq \mathscr{M}[\underset{\sim}{G}_{P_i}, N]$".

First we deal with version 2 and assume that $\mathfrak{p}$ is simple. Choose $y^* \in N$ which codes enough; clearly $\beta' =: f_{i(*)}(\alpha)$ belongs to N. So $f_{i,i(*)}(\beta') \le \beta$, hence by the induction hypothesis there are q', Y' such that: $p \restriction i(*) \le q, q \le q' \in P_{i(*)}, q'$ is $(N, P_{i(*)})$-generic, $Y' \subseteq Y, Y' \in D_{\beta'}(N)$ and $q' \Vdash$ "$Y' \subseteq \mathscr{M}[\underset{\sim}{G}_{P_{i(*)}}, N, y^*]$".

Next, we apply clause (c) in the Definition 4.1 for $i(*)$, so there are q'', Y'' such that $p \le q'', q' \le q'' \in P_{i(*)+1} = P_j, q''$ is (N, P_j)-generic, $Y'' \subseteq Y', Y'' \in D_\alpha(N)$ and $q'' \Vdash$ "$Y \subseteq \mathscr{M}[\underset{\sim}{G}_{P_j}, N, y^*]$".

The proof for version 1 is similar.

Clause $(d)'$:

Recall that we have demanded $f_{i,j}(f_{i,j}(\alpha)) = f_{i,j}(\alpha)$ (see $\boxtimes$ at the beginning of the proof).

Let $N_0, N_1, G^*, \alpha, \beta, i, j, p, k, q_\ell$ (for $\ell < k$), G^ℓ (for $\ell < k$) and G^* be as in the assumption of clause (d) (from Definition 4.1) be given.

Without loss of generality $i < j, i < i(*), j = i(*) + 1$ (as in the proof of clause (c) the other cases are trivial).
First, choose G^{**} for $G^*, \alpha, \beta, i, i(*), G^{**} \in N_1$. For each $\ell < k$, if for some $s_\ell \in G^\ell$ we have

$$s_\ell \Vdash_{P_i} \text{"in } P_{i(*)}/\underset{\sim}{G}_{P_i} \text{ there is an upper bound to } G^{**}\text{"}$$

then (possibly increasing s_ℓ, recalling G_ℓ is generic over N^*) there are $s_\ell \in G^\ell, r_\ell \in P_{i(*)} \cap N_1$ such that s_ℓ forces that r_ℓ is an upper bound to G^{**} and without loss of generality $r_\ell \restriction i \leq s_\ell$. Now without loss of generality $[G_{\ell_1} = G_{\ell_2} \Rightarrow s_{\ell_1} = s_{\ell_2}]$ and $[G_{\ell_1} \neq G_{\ell_2} \Rightarrow s_{\ell_1}, s_{\ell_2}$ incompatible]. Now choose $r \in P_{i(*)} \cap N_1$ with domain $\subseteq i(*) \backslash i$ as follows: $\mathrm{Dom}(r) = \bigcup\limits_{\ell < k} \mathrm{Dom}(r_\ell) \backslash i$ and $r(\alpha)$ is $r_\ell(\alpha)$ if $s_\ell \in \underset{\sim}{G}_{P_i}, \ell < k$ and is $\emptyset_{P_\alpha}$ if this occurs for no ℓ. Renaming $r \in N_i \cap P_{i(*)}$, $\mathrm{Dom}(r) \subseteq i(*) \backslash i$ and $s_\ell \in G^\ell, r_\ell = s_\ell \cup r$ is above G^{**} in $P_{i(*)}$. Let $\beta_\ell = f_{i,j,1+\ell}(\alpha)$ for $\ell \leq k$. We choose by induction on $\ell \leq k$ the objects Y_ℓ, q'_ℓ, M_ℓ such that

$$Y_0 = Y$$

$$M_0 = N_1$$

$$N_0 \in M_{\ell+1}$$

$$M_{\ell+1} \in M_\ell \cap \mathscr{E}_{\beta_{k-\ell}}$$

$$Y_{\ell+1} \subseteq Y_\ell$$

$$Y_\ell \in D_{\beta_\ell}(M_\ell)$$

$$M_{\ell+1} \in Y_\ell$$

$$q_\ell \leq q'_\ell \in P_{i(*)}$$

$$q'_\ell \text{ is } (M_{\ell+1}, P_{i(*)})\text{-generic}$$

$$q'_\ell \text{ forces a value to } \underset{\sim}{G}_{P_{i(*)}} \cap M_{\ell+1}$$

$$q'_\ell \text{ is } (N_0, P_{i(*)})\text{-generic}$$

$$q'_\ell \Vdash \text{“}Y_{\ell+1} \subseteq \mathscr{M}[\underset{\sim}{G}_{P_{i(*)}}, M_\ell]\text{”}$$

$$q'_\ell \restriction i \in G^\ell.$$

(In the older version we have to increase $\beta\,\omega$ times if k not given, arriving to a fixed point.)

No problem, now apply clause (d) of the definition for $i(*), N_0, M_k$, $\langle q'_\ell : \ell < k\rangle, Y_k, G^{**}$ and get G^{***} as required.

Comment: The Y transfers information between the various generics. In [Sh:f, Ch.V], in the first proof after ω steps we are lost, but having the common tower of models help us.

Case 3: $\delta = \ell g(\bar{Q})$ is limit.

Part (1): Follows from clause (d) in part (2) and part (2) is proved below.

Part (2): Let $f_{i,j}$ be fast enough (you can collect the demands used below). Clause $(b)'$ is obvious. We first prove clause $(d)'$ and later $(c)'$.

Clause $(d)'$: So again without loss of generality $i < j = \delta$.
Let $N_0, N_1, p, G^*, \alpha, \beta, k < 1 + \kappa$ and G_ℓ, q_ℓ for $\ell < k$ be as in the assumption of clause (d) in Definition 4.1 but for $i < j$.
We for simplicity use $\kappa = \aleph_0$. We choose $\gamma \in N_0, \alpha < \gamma < \beta$ such that γ is larger enough, in particular: $i \le i' < j' < j \Rightarrow f_{i',j'}(\gamma) = \gamma$.
Choose $\langle i_m : m < \omega\rangle \in N_1$ such that:

$$i_0 = i$$

$$i_m < i_{m+1}$$

$$\sup_{m<\omega} i_m = \sup(j \cap N_0).$$

Choose $y^* \in N \cap \mathscr{H}(\chi_\gamma)$ coding enough. We choose $M_0, M_1, M_2, M_3, M_4 \in N_1 \cap \mathscr{E}_\gamma \cap Y$ such that

$$N_0 \in M_0 \in M_1 \in M_2 \in M_3 \in M_4$$

$$Y \cap M_m \in D_\gamma(M_m) \text{ for } m < 5.$$

Choose $q'_\ell \in G^\ell \cap M_4$ above $G^\ell \cap M_3$ so
q'_ℓ is (M_0, P_i)-generic,(M_1, P_i)-generic,(M_2, P_i)-generic, (M_3, P_i)-generic.
Let $\langle \mathscr{I}_m : m < \omega \rangle \in M_0$ list the dense open subsets of P_j from N_0.

Explanation. Now we shall use the diagonal argument, choose $\underset{\sim}{G}_P \cap N_0, p_m \in P_{i_m} \cap N_0, r_m \in P_{i_n}$ as usual, using in the m-th step $f_{i_m,i_{m+1}}+$ relevant $(d)'$ things. We fulfill the above in M_4, so in the end can find a solution in N_1, by using a canonical construction, e.g. each time the $<^*_\chi$-choice.

But to carry this we need to have finitely many candidates for $\underset{\sim}{G}_{P_{i_m}} \cap M_0$ with a common Y_m.
(Note: if $Q \times \ldots \times Q$ is proper, like in Ch.XVIII,§2 we can get such a common Y_m, i.e. the present result is stronger for this stage.)
To get this in the inductive step, we need in step $m-1$ that for M_1 we just have finitely many candidates for $\underset{\sim}{G}_{P_{i_m}} \cap M_1$, and in turn to get this in the step $m-1$ we use that in step $m-2$ for M_2 we have: from every maximal antichain we choose a finite subset. To get this we use that for M_3 we just ask $M_3[\underset{\sim}{G}_{P_{i_{m-3}}}] \cap V = M_3$. So along the way N_0, M_0, M_1, M_2, M_3 our induction demands go down, but slowly, so that in each step m, advancing for say M_0, we have to preserve less than really knowing $\underset{\sim}{G}_{P_{i_m}} \cap M_0$, and are helped by our demand on M_1, just like in [Sh:f, Ch.XVIII]. So compared to [Sh:f, Ch.V], we have a finite tower.

Returning to the proof we choose by induction on $m < \omega$ the objects r_m, G^*_m, p_m, n_m, $\langle G^\ell_m : \ell < n_m \rangle, Y_m$ such that

- (a) $r_m \in P_{i_m} \cap M_4$
- (b) $\mathrm{Dom}(r_m) \subseteq [i, i_m)$
- (c) $r_{m+1} \restriction i_m = r_m$
  (comment: r_m's act as a generic for N_0)
- (d) $q'_\ell \cup r_m \in P_{i_m}$ is (M_0, P_{i_m})-generic, (M_1, P_{i_m})-generic, (M_2, P_{i_m})-generic, (M_3, P_{i_m})-generic
- (e) if $\ell < k$, $\mathscr{J} \subseteq P_{i_m}$ is dense open and $\mathscr{J} \in M_2$, <u>then</u> for some finite $\mathscr{J} \subseteq \mathscr{I} \cap M_2$, $\mathscr{J}$ is predense above $q'_\ell \cup r_m$
  (like in the proof of ${}^\omega\omega$-bounding)
- (f) $n_m < \omega$, and
  $\ell < n_m \Rightarrow G^\ell_m$ is a subset of $P_{i_m} \cap M_0$ generic over M_0,
  and $G^\ell_m \in M_1$

(g) $G^\ell_{m+1} \cap P_{i_m} \in \{G^\ell_m : \ell < n_m\}$

(h) $n_0 = k, G^\ell_0 = G^\ell \cap M_1$

(i) $q_\ell \cup r_m \Vdash$ "$\underset{\sim}{G}_{P_{i,m}} \cap M_1 \in \{G^\ell_m : \ell < n_m\}$"

(j) G^*_m is a subset of $P_{i_m} \cap N_0$ generic over N_0

(k) $G^*_m \subseteq G^\ell_m$, so $G^*_m \subseteq G^*_{m+1}, G^*_0 = G^*$

(l) $p_m \in P_j \cap N$

(m) $p_0 = p$

(n) $p_m \restriction i_m \in G^*_m$

(o) $p_m \leq p_{m+1} \in \mathscr{I}_m$

(p) $Y_m \subseteq \mathscr{M}[G^\ell_m, M_0, y^*]$

(q) $Y_m \in D_\gamma(M_0)$.

Why is this sufficient? If we can carry out the induction, then without loss of generality the construction belongs to N_1.
So $G^{**} = \{s \in P_j \cap N_0 : \bigvee\limits_{n<\omega} s \leq p_m\}$ is as required, as $q'_\ell =: q_\ell \cup \bigcup\limits_m r_m \in P_j \cap N_1$, and is above G^{**} and $p \leq q'_\ell$.

Induction:

$\underline{m = 0}$: Trivial.

$\underline{m + 1}$: Stage A: Choosing p_{m+1} is trivial, the demands are: $p_{m+1} \geq p_m, p_{m+1} \restriction i_m \in G^*_m$ and $p_{m+1} \in \mathscr{I}_m$.

Stage B: Choosing G^*_{m+1}: apply the induction hypothesis using clause $(d)'$ of what we are proving with $i_m, i_{m+1}, \gamma, f_{i_m,i_{m+1}}(\gamma), N_0, M_1$ here standing for $i, j, \alpha, \beta, N_0, N_1$ there.

Stage C: Let $\{H^\ell_m : \ell < n_{m+1}\}$ list the possibilities of $\underset{\sim}{G}_{P_{i_m}} \cap M_1$ (by clause (e) this exists).
Without loss of generality $H^\ell_m \cap M_0 = G^{h(\ell)}_m$, for some function $h = h_m : n_{m+1} \to n_m$.
We choose $s^\ell_m \in P_{i_{m+1}} \cap M_1$, above G^*_{m+1}, such that $s^\ell_m \restriction i_m \in G^{h(\ell)}_m$. Now repeat the argument of the successor stage of shrinking Y (but now we have a fixed γ!).

So we can find $t^\ell_m \in P_{i_{m+1}} \cap M_1$, above $s^\ell_m, t^\ell_m \restriction i_m \in H^\ell_m, t^\ell_n \Vdash$ "$\underset{\sim}{G}_{P_{i_{m+1}}} \cap M_0 =: G^\ell_{m+1}$" such that

$$Y_{m+1} =: \bigcap_{i<n_{m+1}} \mathscr{M}[G^\ell_{m+1}, M_0, y^*] \in D_\gamma(M_0).$$

The rest is as in §1 so we have finished proving clause $(d)'$ in the case $\ell g(\bar{Q})$ is a limit ordinal (which is last).

Clause $(c)'$: Again, without loss of generality $i < j = \delta$. So assume $f_{i,j}(\alpha) \le \beta, \{i, j, \alpha, \beta\} \in N^* \in \mathscr{E}_\beta, q$ is (N^*, P_i)-generic, $Y^* \in D_\beta(N^*), q \in P_\alpha$ and $q \Vdash$ "$Y^* \subseteq \mathscr{M}[\underset{\sim}{G}_{P_i}, N^*, y^*]$" are given. We prove the desired conclusion by induction on α. For each α, we would like to simulate a play of $\eth_{\alpha,\beta}(N^*)$, supplying the challenger with a strategy. For this we apply the proof of clause $(d)'$.
Choose $N_0, N_1, M_0, \ldots, M_4, q_0, G_0, G^*$ (and $k = 1$) as there for some $\alpha' < \gamma' < \beta'$ as there such that $\beta < \alpha'$ such that $N^*, q, Y^* \in N_0$ (easy to find).

Explanation: Note: as $\mathfrak{p}$ is simple, we can use $N \prec (\mathscr{H}(\chi), \in), N \cap \mathscr{H}(\chi_\beta) \in \mathscr{E}_\beta$ with no mention - there were other places we could have said so. During the construction this time we demand $p_m \in N^* \cap P_j$, so a generic for N_0 is not necessarily created. But still $p_m \le p_{m+1}, p_m \restriction i_m \in G^*_m$. Now p_m will be played by the chooser. Now $g(1+\alpha)$ will be a fixed point of f_{i_m, i_m+1}. So we can add the demand $N^*[G^*_m \cap N^*] \cap V = N^*$, i.e. G^*_m is generic over N^* and

$$\mathscr{M}[G^*_m \cap N^*, N^*, y^*] \in D_{\delta(1+\alpha)}(N^*).$$

The challenger chooses

$$X_{m+1} = \mathscr{M}[G^*_m \cap N^*, N^*, y^*] \cap \bigcup_{\xi<\gamma(1+\alpha)} \mathscr{E}_\xi \cap \{M : p_n \in M\} \in N_0.$$

Now the chooser chooses α_m, β'_m and the challenger chooses $\beta_n \ge \beta'_m, f_{i_m, i_{m+1}}(\alpha)$ in $N^*_0 \cap j$ and the chooser chooses M^*_m such that $p_m \in M_{m+1}$.

Comment: The game was defined with this point in mind.
Now the chooser chooses $Y_n \in D_{\beta_n}(M_0), Y_n \subseteq X_n \cap M_0, Y_n \in N^*_0$ (check definition of game!)
Now we play Z_n for the challenger as follows:
there is $p_{m+1} \ge p_m, (M^*_{m+1}, P_{i_{m+1}})$-generic
$p_{m+1} \restriction i_m \in G^*_m$
and p'_{m+1} forces $\underset{\sim}{G}_{P_{i_{m+1}}} \cap N^*$ and

forces $Z_n = Y_n \cap \mathscr{M}[\underset{\sim}{G}_{P_{i_{m+1}}}, M^*_m, y^*] \in D_{\alpha_n}(M^*_m)$.

As the challenger does not have a winning strategy, there is a play he wins, giving $\bigcup\limits_{n<\omega} G^*_n \cap N^*_0$ with a bound.

Continuation of the Proof: Choose also $\langle i'_m : m < \omega\rangle \in N_0$ such that $i_m \in N^*, i_0 = i, i_m < i_{m+1}$, $\sup\{i_m : m < \omega\} = \sup(N^* \cap j)$ and this time we let $\langle \mathscr{I}'_m : m < \omega\rangle$ list the dense open subsets of P_j from N^*. For $\mathbf{m} < \omega$ let $\mathscr{T}_{\mathbf{m}}$ be the set of the finite sequences $\mathfrak{x}$ from M_4 coding $\langle r_{\mathfrak{x},m} : m \le \mathbf{m}\rangle, \langle G_{\mathfrak{x},m} : m \le \mathbf{m}\rangle, \langle p_{\mathfrak{x},m} : m \le \mathbf{m}\rangle, \langle n_{\mathfrak{x},m} : m \le \mathbf{m}\rangle, \langle G^\ell_{\mathfrak{x},m} : \ell \le n_{\mathfrak{x},m}, m \le \mathbf{m}\rangle, \langle Y_{\mathfrak{x},m} : m \le \mathbf{m}\rangle$ and also $(X_{\mathfrak{x},m}, \alpha_{\mathfrak{x},m}, \beta'_{\mathfrak{x},m}, \beta_{\mathfrak{x},m}, M_{\mathfrak{x},n}, y'_{\mathfrak{x},m}, M'_{\mathfrak{x},m}, y_{\mathfrak{x},m})$ for $m \le \mathbf{m}$ and $Z_{\mathfrak{x},m}$ for $m < \mathbf{m}$ satisfying:

clauses (a)-(k),(m),(n),(p),(q) from the proof of (d)$'$ above and

- $(l)'$ $p_m \in_j \cap N^*$
- $(o)'$ $p_m \le p_{m+1} \in \mathscr{I}'_n$
- $(r)'$ $r_{\mathfrak{x},m}$ is (N^*, G_{i_m})-generic for $m \le \mathbf{m}$
- $(s)'$ $\langle (X_{\mathfrak{x},m}, \alpha_{\mathfrak{x},m}, \beta'_{\mathfrak{x},m}, \beta_{\mathfrak{x},m}, M_{\mathfrak{x},m}, Y_{\mathfrak{x},m}, M'_{\mathfrak{x},m}, Z_{\mathfrak{x},m'}) : m \le \mathbf{m}\rangle$ belongs to N and is an initial segment of a play of the game $\partial'_{\alpha,\beta}(N^*, \mathfrak{p})$ or just $\partial'_{\alpha,\beta}(N^*, N, \mathfrak{p})$, note that in the $\mathbf{m}$-th move the challenger has not yet chose $Z_{\mathfrak{x},\mathbf{m}}$, (see clause (e) of Definition 2.2(1)
- $(t)'$ $Z_{\mathfrak{x},m} \subseteq Y_{m+1}$
- $(u)'$ $y_{\mathfrak{x},m}$ code $p_m, \langle i_m : m < \omega\rangle$
- $(v)'$ $f_{i_m,i_{m+1}}(\alpha_m) \le \beta'_m$ for $m \le \mathbf{m}$.

Let $\mathfrak{x} \triangleleft \mathfrak{y}$ has the natural meaning for $\mathfrak{x} \in \mathscr{T}_{\mathbf{m}_1}, \mathfrak{y} \in \mathscr{T}_{\mathbf{m}_2}, \mathbf{m}_1 < \mathbf{m}_2$.
Note

- $\boxtimes_1$ $\mathscr{T}_{\mathbf{m}} \subseteq N$ for $\mathbf{m} < \omega$
- $\boxtimes_2$ $\mathscr{T}_0 \ne \emptyset$
- $\boxtimes_3$ if $\mathbf{x} \in \mathscr{T}_{\mathbf{m}}$, then considering the game $\partial_{\alpha,\beta}(N^*, \mathfrak{p}), \mathfrak{x}$ is an initial segment of a play of it (see clause $(s)'$ above).

Assume

- $(*)$ $M'_{\mathbf{m}} \in Y_{\mathfrak{x},\mathbf{m}} \cap \mathscr{E}_{\alpha_{\mathbf{m}}} \cap (M_{\mathfrak{x},\mathbf{m}} \cup \{M_{\mathfrak{x},\mathbf{m}} \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\mathbf{m}})\}$ satisfying $y_{\mathbf{m}}, y'_{\mathbf{m}} \in M'_{\mathbf{m}}$ and $Z_{\mathbf{m}} \subseteq \mathscr{D}_{\alpha_{\mathfrak{x},m}}(M'_m), Z_{\mathbf{m}} \subseteq Y_{\mathbf{m}}$ (hence $Z_{\mathbf{m}} \subseteq X_{\mathfrak{x},\mathbf{m}}$) and $X_{\mathbf{m}+1} \in D_\beta(N^*) \cap X_{\mathfrak{x},\mathbf{m}}$ such that $Z_{\mathbf{m}} \subseteq X_{\mathbf{m}+1}$ and $\alpha_{\mathbf{m}+1} \in \alpha \cap N^*$ and any $\beta'_{\mathbf{m}+1} \in \beta \cap N^* \backslash p_{i_{\mathbf{m}+1},i_{\mathbf{m}_r}}(\alpha_{\mathbf{m}+1}), y'_{\mathbf{m}+1} \in N \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_{\mathbf{m}+1}})$ and any $\beta_{\mathbf{m}} \in \beta \cap N \backslash \beta'_n \backslash \alpha_n$ and $M_{\mathbf{m}+1} \in X_{\mathbf{m}+1} \cap \mathscr{E}_{\beta_{\mathbf{m}+1}}, y_{\mathbf{m}+1} \in M_{\mathbf{m}+1} \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_{\mathbf{m}}})$ satisfying $y'_{\mathbf{m}+1} \in$

$M_{\mathbf{m}+1}$ and $Y_{\mathbf{m}+1} \in N \cap D_{X_{\mathbf{m}+1}}(M_{\mathbf{m}+1})$ and any $M'_{\mathbf{m}+1} \in Y_{\mathbf{m}+1} \cap \mathscr{E}_{\alpha_n} \cap (M_{\mathbf{m}+1} \cup \{M_{\mathbf{m}+1} \cap \mathscr{H}(\chi^{\mathfrak{p}}_{\alpha_m})\}$ satisfying $y_{\mathbf{m}+1}, y'_{\mathbf{m}+1} \in M'_{\mathbf{m}+1}$.

Then

$(**)$ there is $\mathfrak{y} \in \mathscr{T}_{\mathbf{m}+1}$ such that $\mathfrak{x} \triangleleft \mathfrak{y}$ and $(Z_{\mathfrak{y},\mathbf{m}}, X_{\mathfrak{y},\mathbf{m}+1}, \alpha_{\mathfrak{y},\mathbf{m}+1}, \beta'_{\mathfrak{y},m+1}, y'_{\mathfrak{y},\mathbf{m}+1}, \beta_{\mathfrak{y},\mathbf{m}+1}, y_{\mathfrak{y},\mathbf{m}+1}, M_{\mathfrak{y},\mathbf{m}+1}, Y_{\mathfrak{y},\mathbf{m}+1}, M'_{\mathfrak{y},\mathbf{m}+1})$ is equal to $(Z_{\mathbf{m}}, X_{\mathbf{m}+1}, \alpha_{\mathbf{m}+1}, \beta'_{\mathbf{m}+1}, y'_{\mathbf{m}+1}, \beta_{\mathbf{m}+1}, y_{\mathbf{m}+1}, M_{\mathbf{m}+1}, Y_{\mathbf{m}+1}, M'_{\mathbf{m}+1})$.
Why? Because $f_{i_m,i_{m+1}}(\alpha_{\mathbf{m}}) \le \beta_{\mathbf{m}}$, hence we know that $P_{i_{\mathbf{m}+1}}/P_{i_{\mathbf{m}}}$ is $(\mathfrak{p}, \alpha_n, \beta_n)$-proper and let $G_{i_{\mathbf{m}}} \subseteq P_{i_m}$ be generic over $V, r_{\mathbf{m}} \in G_{i_{\mathbf{m}}}$ to the model $M'_{\mathfrak{x},\mathbf{m}}$ and the set $Y_{\mathfrak{x},m}$. So now we can describe a strategy for the challenger in the game $\eth_{\alpha,\beta}(N^*, \mathfrak{p})$ (or $\eth'_{\alpha,\beta}(N^*, N, \mathfrak{p})$) delaying his choice of M'_m, Z_m to the $(m+1)$-th move, he just chose on the side $\mathfrak{x}_m \in \mathscr{T}_m$ which "code" they play so far, and preserve $\mathfrak{x}_m \triangleleft \mathfrak{x}_{m+1}$.

By $\boxtimes_3$ this is O.K. - all possible choices of the chooser are allowed, that is this gives a well defined strategy for the challenger (will he have some free choice, this does not hurt). But the chosen does not lose the game, so there is such a choice $\langle \mathfrak{x}_{\mathbf{m}} : \mathbf{m} < \omega \rangle$ with $\cup\{(M'_{\mathfrak{x}_{m+1},\mathbf{m}}\} \cup Y_{\mathfrak{x}_{m+1},\mathbf{m}} : \mathbf{m} < \omega\} \in D_\alpha(N)$ and $\bigcup_{m<\omega} r_m$ is as required.

* * *

The adaptation for the proof when $\kappa = 2$ should be clear.

Proof. Similar to the proof of 1.9, with some changes. We choose $\langle j_n : n < \omega \rangle \in M_0$ such that $j < j_n < j_{n+1} < h^*(j,i), h^*(j_n, i_m) < j_{n+1}$

(l) n_m is a power of 2, say $2^{n^*_m}$ and so we can rename $\{G^\ell_m : \ell < n_m\}$ as $\{G^\eta_m : \eta \in 2^{n^*_m}\}$

(m)

- (α) $M_\eta \in M_1$ for $\eta \in {}^{(n^*_m \ge)}2$
- (β) $M_{<>} = N_0$
- (γ) $M_\eta \in M_{\eta\char`^\langle 0\rangle} \cap M_{\eta\char`^\langle 1\rangle}$
- (δ) $\eta \triangleleft \nu_1 \in {}^{n^*_m}2, \eta \triangleleft \nu_2 \in {}^{n^*_m}2 \Rightarrow G^{\nu_1}_m \cap M_\eta = G^{\nu_2}_m \cap M_\eta$ so we call it K^η_m
- (ε) $M_{\eta\char`^\langle 0\rangle} = M_{\eta\char`^\langle 1\rangle}$ when $\eta \in {}^{n^*_m}2$ call it N_η
- (ζ) $N_\eta \in \mathscr{E}_{\ell g(\eta)}$ for $\eta \in {}^{(m^*_m >)}2$
- (η) $Y_\eta = \mathscr{M}[K_{\eta\char`^\langle 0\rangle}, N_\eta] \cap \mathscr{M}[K_{\eta\char`^\langle 1\rangle}, N_\eta] \in D_{j_{\ell g(\eta)}}(N_\eta)$.

Discussion: We may be interested in non-proper forcing, say semi-proper and UP ones (see [Sh:f, Ch.X,XI,XV] and [Sh 311]). Here the change from (reasonable) parameter $\mathfrak{p} = \mathfrak{p}^V$ to $\mathfrak{p}^{V[G]}$ is more serious as $\{N \cap \chi_\alpha : N \in \mathscr{E}^{\mathfrak{p}^{V[G]}}\}$ is in general not equal to $\{N \cap \chi_\alpha : N \in \mathscr{E}^{\mathfrak{p}}_\alpha\}$. We intend to spell it out in [Sh 311].

We may be interested in combining this work with [Sh 587], [Sh:F259]. Intend to deal with it later (see also [Sh 655]).

§5 Problematic forcing

5.1 Discussion: 1) In the examples the "barely adding reals but not so" may occur on some stationary $\mathscr{E} \subseteq [\chi^*]^{\aleph_0}$ and otherwise just properness is asked. We do not bother to do this in the examples.
2) We may like to put the present lemmas and [Sh:f, Ch.XVIII,§2] and more together. The way is clear, we concentrate on $\kappa \in \aleph_0$.

So instead of a "tower" with six levels we have one with $n(*)+1$ levels.

5.2 Definition. 1) We say $\bar{Q}$ is an $\mathfrak{p} - NNR^0_{\aleph_0,\bar{P}r,\bar{\Xi}}$-iteration for $\mathfrak{p}$ is:

- (a) $\bar{Q}$ is a CS iteration of proper notions forcing, $\mathfrak{p}$-proper
- (b) forcing with $\mathrm{Lim}(\bar{Q}) = P_{\ell g(\bar{Q})}$ does not add reals
- (c) $\bar{P}r = \langle Pr_\ell : \ell < n(*)\rangle, \bar{\Xi} = \langle \Xi_\ell : \ell \le n(*)\rangle, \Xi_\ell \subseteq \ell g(\mathfrak{p}) \times \ell g(\mathfrak{p})$
- (d) $Pr_\ell(N,\bar{G},P)$ implies: P a forcing notion, N is a countable elementary submodel of $(\mathscr{H}(\chi),\in), \bar{G} = \langle G_m : m < k\rangle, k < \omega, G_m \subseteq N \cap P$ is generic over N
- (e) if $\ell = 0, Pr_\ell(N,\bar{G},P)$, then $\bigwedge\limits_\ell G_\ell = G_0$
- (f) if $\ell = n(*)-1$, then $Pr_\ell(N,\bar{G},P)$, iff the demand in clause (d) holds
- (g) if $\mathrm{Pr}_\ell(N,\bar{G},P)$ and $\bar{G} \triangleleft \bar{G}', \ell g(\bar{G}')$ finite and $\mathrm{Rang}(\bar{G}) = \mathrm{Rang}(\bar{G}')$ then $Pr_\ell(N,\bar{G}',P)$
- (h) $Pr_0(N,\bar{G},P)$ holds iff (the condition in clause (c) and) $\bigwedge\limits_\ell G_\ell = G_0$
- (i) if $\ell < n(*), i < j \le \ell g(\bar{Q}), N_0 \prec N_1 \prec (\mathscr{H}(\chi),\in), \mathfrak{p} \in N_0, N_0, N_1$ are countable and for some $(\alpha,\beta) \in \Xi_\ell \cap N_0$ we have $N_0 \cap \mathscr{H}(\chi^{\mathfrak{p}}_\alpha) \in \mathscr{E}_\alpha, N_1 \cap \mathscr{H}(\chi^{\mathfrak{p}}_\beta) \in \mathscr{E}_\beta, Pr_\ell(N_0,\bar{G},P_i), Pr_{\ell+1}(N_1,\bar{H},P_i), \ell g(\bar{G}) = \ell g(\bar{H}), G_k \subseteq H_k, \bar{p} = \langle p_k : k < \ell g(\bar{G})\rangle, p_\ell \in N_0 \cap P_j, p_k \restriction i \in G_k, \ell = 0 \Rightarrow p_k = p_0$, then we can find $\bar{G}^+ = \langle G^+_k : k < \ell g(\bar{G})\rangle \in N_1$ such that $Pr(N_0,\bar{G}^+,P_j)$ and $p_k \in G^+_k$
- (j) assume $Pr_\ell(N,\bar{G},P_i), i_n < i_{n+1}, i_n \in N, i_0 = i, \sup_{n<\omega} i_n = \sup(N \cap j), (\alpha,\beta) \in N \cap \Xi_\ell, N' = N \cap \mathscr{H}(\chi^{\mathfrak{p}}_\alpha) \in \mathscr{E}_\alpha$, then in the following game the Pr^+_ℓ-player has a winning strategy (or just does not lose).

Before the n-th move $\bar{G}^n$ such that $Pr_\ell(N,\bar{G}^n,P_{i_n})$ is chosen $\ell g(\bar{G}^n) = \ell g(\bar{Q})$ with $\bar{G}^0 = \bar{G}$. In the n-th move the challenger chooses $\bar{p}^n = \langle p^n_k : k < \ell g(\bar{G})\rangle, p^n_k \in N_0 \cap P_{in+1}, P^n_k \restriction i_n \in G^n_k, \ell = 0 \Rightarrow p^n_k = p^n_0$ and then the chooser chooses $\bar{G}^{n+1}$ is above such that $p^n_k \in G^{n+1}_k$. In the end of the play the chooser wins if $Pr_\ell(N,\bar{G}^*,P_j)$ where $G^*_k = \{p \in N \cap P_j : \text{for every } n, p \restriction i_n \in G^n_k\}$.

<u>5.3 Comment</u>: 0) Of course, we can also use a reasonable parameter for induction.
1) In [Sh:f, Ch.XVIII,§2] for NNR^2_κ we can use weaken clause (5): t a tree with 2 branches if $\bar{Q} \in N \prec (\mathscr{H}(\chi), \in)$, if $i^* < \alpha, P_{i^*,i} =: \{(q, p_0, p_1) \in P_i : p_0 \restriction i^* = p_1 \restriction i^*\}$, then for $i < j \leq \delta, i$ non-limit we have: $P_{i^*,i}/P_{i^*,j}$ is $\mathscr{E}_2$-proper.

In light of the previous proofs this is straight.
2) We can replace $D_\alpha(N)$ by $D_{\alpha,i,j}(N)$ for $i < j \leq \delta^*, \delta^*$ the supremum of the length of the iterations we consider.

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